The principle of least energy-dissipation rate and the possibility of its application to turbulent flow in a plane channel

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Variational principles are used extensively in the mechanics of continuous media. Equations of motion or of equilibrium for the continuous medium under consideration are obtained from the condition for the minimum of an appropriate functional. Apparently, Malkus[2] first used a variational principle to close the Reynolds system by applying the so-called principle of the greatest rate of dissipation of the energy at perturbations in turbulent flow and imposing the requirement of neutral stability on the average-velocity profile. Gol'dshtik[5] proposed a principle of greatest stability of the averaged motion.

In the case of slow motions in hydrodynamics, when inertial terms are neglected, a variational principle can also be formulated. Helmholtz[1] did this and called it the "principle of least rate of energy dissipation." The Helmholtz principle is valid for all slow stationary flows of a viscous fluid. However, there are flows in which there are certainly no inertial terms although the flows are far from slow: for instance, plane-parallel flow in a two-dimensional channel with parallel walls, flow in a circular pipe, etc. It is easy to show that the principle of least energy-dissipation rate is completely valid for them also, since there are no inertial terms.

It is possible to try to apply this principle to turbulent flow in a plane channel too, because inertial terms in the averaged flow are also absent here. The inertial interactions affect only the turbulent stresses. In this case the principle of least dissipation will mean that the t o t a l r a t e of dissipation in turbulent flow should be m i n i m a l, i.e., of all the kinematically possible turbulent flows only that for which the total average rate of energy dissipation is least will actually be realized.

1. Plane-parallel, laminar, stationary flow of a viscous fluid in a plane channel is described by the equations

$$\nu \frac{d^2\bar{u}}{dy^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} = \text{const} = -A; \qquad (1.1)$$

$$\frac{\partial u}{\partial x} = 0; \qquad (1.2)$$

where ν is the viscosity, ρ is the fluid density, u is the only nonzero velocity component, and p is the pressure. The coordinate system is shown in Fig. 1. When the pressure gradient is given, the characteristic velocity of this flow is $u^* = \sqrt{|A|H}$, and is called the dynamic velocity.

For definiteness, let A > 0. The solution of (1.1) in dimensionless form is

$$v = \mathrm{Re}/2(1-\eta^2), \qquad (1.3)$$

where $vu^* = u$, $y = H\eta$ and the Reynolds number is calculated with respect to the dynamic velocity $\mathrm{Re} = u^*H/\nu$.

The mean dissipation rate equals

$$D^m = \frac{u^{*3}}{H}\int_0^1 \left(\frac{dv}{d\eta}\right)\frac{1}{2\mathrm{Re}}\,d\eta = \frac{u^{*3}}{H}\,\frac{\mathrm{Re}}{6}. \qquad (1.4)$$

Finally, it is easy to show that (1.1) is the Euler-Lagrange equation that should be satisfied by the function minimizing the functional

$$J = \frac{1}{LH}\int_0^L\int_{-H}^{H} \frac{\nu}{2}\left(\frac{du}{dy}\right)^2 dy\,dx, \qquad (1.5)$$

when the constraint (1.2) is imposed on u; the pressure p is the Lagrange multiplier for this constraint.

Since (1.5) is none other than the mean rate of dissipation per unit length and area of the channel section, it can be asserted that the laminar steady-state flow of a viscous incompressible fluid is subject to the principle of least dissipation.

2. Experiments show that for a Reynolds number greater than some critical value Re_{cr}, the flow mode will cease to be laminar and will become turbulent. It is known that in this case the Reynolds equation reduces to one equation[6]

$$\nu\frac{d^2\overline{u_x}}{dy^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{d\overline{u'_x u'_y}}{dy}, \qquad (2.1)$$

where again $(1/\rho)\partial p/\partial x = -A$ (the coordinate system is the same as in Fig. 1). The continuity equation yields

$$\frac{\partial \overline{u_x}}{\partial x} = 0. \qquad (2.2)$$

Averaged quantities in (2.1) and (2.3) have a bar above them, while the pulsations have a prime. Any of the known averaging methods[6] can be used for the averaging.

Equation (2.1) contains two functions, the average velocity $\bar{u}_x$ and the mixed second moment $\overline{u'_x u'_y}$. Since the former is an even function of y and the latter is odd, we can limit ourselves to just the interval $y \in [0, H]$. Without limiting the generality, it can be assumed in this interval that $\overline{u'_x u'_y} \geq 0$ (Ref. 6). After introducing the dimensionless variables

$$v = \overline{u_x}/u^*, \quad \eta = y/H, \quad \sigma^2 = \overline{u'_x u'_y}/u^{*2}$$

integration of (2.1) with the boundary conditions $v(1) = 0$, $v'(1) = 0$ and $\sigma(0) = \sigma(1) = 0$ yields

$$\frac{1}{\mathrm{Re}}\frac{dv}{d\eta} = -\eta + \sigma^2. \qquad (2.3)$$

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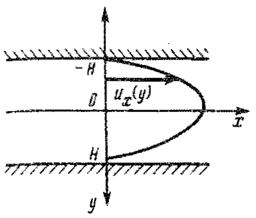

FIG. 1

In order to use the principle formulated above, the connection between the energy-dissipation rate and the Reynolds stress or some other stress must be given. On the basis of dimensional analysis, we can adopt, e.g., formulas of the Rott–Kolmogorov type

$$\epsilon = \frac{2}{3}\,\frac{(u'_x u'_y)^{3/2}}{l'} = \frac{u^{*3}}{H}\cdot\frac{2}{3}\,\frac{\sigma^3}{l}, \qquad (2.4)$$

where ε is the dissipation rate, and l' is the dimensional and l the dimensionless mixing path (the value of l is usually directly proportional to the distance from the wall). This formula yields satisfactory results principally in the flow core, where the turbulence can be assumed to be almost homogeneous and isotropic.

3. As has been remarked, there are no inertial interactions in the averaged flow, and, in some sense, this is analogous to laminar flow in a plane channel, where there were also no inertial terms in the equations. Continuing analogously, it can be assumed that the principle of least dissipation is valid also for this "inertia-free" turbulent flow, namely:

For a given pressure gradient in a plane channel, that turbulent (or laminar) flow will be realized for which the mean energy-dissipation rate per unit length and unit cross section of the channel is least.

It must be emphasized that what we are speaking of is the minimal dissipation for a given drop. If we do not have this requirement (for instance, for a given flow rate), then the dissipation in turbulent flow will be very much greater than in laminar flow. Then it already turns out that it is impossible, generally, to expect the appearance of a turbulent regime.

The mean rate of turbulent-flow dissipation is

$$D_T = \frac{u^{*3}}{H}\int_0^1\left[\frac{1}{2\mathrm{Re}}\left(\frac{dv^2}{d\eta}\right)^2 + \frac{2}{3}\,\frac{\sigma^3}{l}\right]d\eta; \qquad (3.1)$$

here the first term is the dissipation of the averaged flow while the second is that of turbulent vortices. It is easy to seek the minimum of this functional under the constraint (2.3): $dv/d\eta$ from (2.3) must be substituted into (3.1). The problem then reduces to finding the minimum of the functional

$$J_1 = \int_0^1 [\tfrac{1}{2}\mathrm{Re}\,(\sigma^2-\eta)^2 + \tfrac{2}{3}\,\frac{\sigma^3}{l}]\,d\eta. \qquad (3.2)$$

The Euler–Lagrange equation for this functional

$$\mathrm{Re}\,\sigma(\sigma^2-\eta) + \sigma^2/l = 0$$

has three solutions. The trivial solution $\sigma \equiv 0$ corresponds to the laminar regime, and, of the two nontrivial solutions, we choose that which satisfies the condition

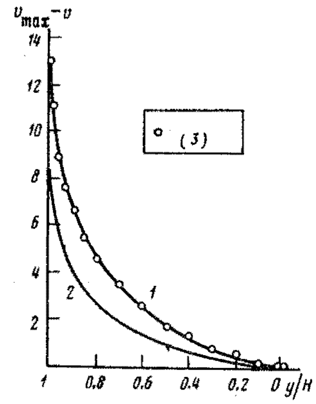

FIG. 2

$\sigma(0) = 0$. Using the notation $1/(2\mathrm{Re}) = m$, we obtain

$$\sigma = \eta/(m/l + \sqrt{\eta + (m/l)^2}). \qquad (3.3)$$

Substituting (3.3) into (2.3) results in the equation

$$\frac{1}{\mathrm{Re}}\,\frac{dv}{d\eta} = -\frac{2\eta}{\sqrt{1+\eta(l/m)^2}+1}; \qquad (3.4)$$

in this equation $m \ll 1$, because turbulent flow exists only if $\mathrm{Re} \geq \mathrm{Re}_{cr} \gg 1$. Then, except for a region of thickness of order $O(m^2)$ around the channel axis, $\eta/m^2 \gg 1$ is satisfied and

$$\frac{1}{\mathrm{Re}}\,\frac{dv}{d\eta} = -\frac{1}{\mathrm{Re}}\,\frac{\sqrt{\eta}}{l} + O(m), \qquad (3.5)$$

but this is none other than the dimensionless Prandtl relation[4]

$$l^2\left(\frac{dv}{d\eta}\right)^2 = \eta = \tau,$$

in which it has already been taken into account that $\tau = \eta$.

Thus, it turns out that the necessary condition for the minimum dissipation rate [when, of course, it is subject to (2.4)] yields the Prandtl formula and the function l is just the Prandtl mixing path. It is known that this latter formula agrees very well with experiment, provided that the empirical function l is chosen felicitously. The simplest assumption about the mixing path is given by Prandtl (Ref. 4): $l = \kappa(1-\eta)$, where $\kappa = 0.4$. Then,

$$v_{max} - v = -\frac{1}{\kappa}\,[2\sqrt{\eta} - \ln(1+\sqrt{\eta}) + \ln(1-\sqrt{\eta})]. \qquad (3.6)$$

Results obtained on the basis of this formula are compared with the data of Nikuradse[3] (see Fig. 2, curve 2). Much better agreement with experiment is displayed if we select $l = {}^1/_3\kappa(1-\eta^3)$, where the coefficient is selected so that $l \approx \kappa(1-\eta)$ for $\eta \to 1$ (Ref. 7). Then,

$$v_{max} - v = \frac{1}{\kappa}\ln\frac{1+\eta^{3/2}}{1-\eta^{3/2}}, \qquad (3.7)$$

which yields curve 1 in Fig. 2.

Finally, it is easy to show that the dissipation for turbulent motion for sufficiently large Re is indeed less than for laminar motion. If we take (3.7), then the least Re for which turbulent dissipation is less than laminar dissipation will be $\mathrm{Re}_{cr} \cong 100$. Then the critical Reynolds number, constructed from the maximum velocity, equals ~ 5000.

The method developed above can also be used for the flow in a circular pipe, and for the boundary layer on a flat plate.

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Structure of turbulence near the core of a vortex ring

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1. The formation and motion of rings has been well known for over a century. It has been established that, together with the ordinarily visible toroidal "core," a much larger volume of the fluid is displaced – the "atmosphere" of the ring. A flow scheme (the Maxwell vortex, according to which the whole vorticity is concentrated in a time toroidal core) has been suggested[1,2] so as to obtain qualitative agreement with experiment. A critical value of the Reynolds number $Re = RU/\nu$, calculated from the initial radius R and velocity U of the vortex ring, can be chosen as the criterion for onset of the turbulence regime. According to the data of Refs. 2 and 3, $Re_{cr} \sim 10^3$. The theory of the motion of turbulent vortex rings[2,3] assumes constancy of the turbulent characteristics over the whole volume of the flow. It is shown here that in the region of the vortex core there exists a region in which turbulence is suppressed.

2. The generator of vortex rings used here is a tube, from which a given volume of fluid is expelled by a piston; all experiments were performed in water, with a range of Re values from $2 \cdot 10^4$ to $4 \cdot 10^4$.

Three variants of the experiments were carried out. In the first experiment the liquid was colored beforehand. The sequence of observed patterns is shown on Fig. 1. Initially the vortex ring is fully colored. The color is then quickly lost, so that after a certain time only the fine toroidal core remains colored.

In the second experiment an initially "clean" vortex was passed through a layer of colored liquid. In this case the ring atmosphere becomes colored. The main part of this color is quickly lost and it becomes obvious that the fine toroidal core corresponding to the colored core in the bottom frame of Fig. 1 is "clean" (Fig. 3). An enhanced dye concentration (a hoop) is observed on the boundary of this core. We draw attention to the fact that the boundaries of the hoop and of the colored liquid in Fig. 1 are sharp.

In the third experiment a strip of color was applied to the inner wall of the generator, near a cut in the tube. After the piston starts to move the particles of the liquid passing near this strip become colored. A spiral, consisting of these particles, is observed in the process of formation of the ring (Fig. 4). The central part of this spiral lies in the core of the vortex formed. In Fig. 2 we show a motion-picture sequence of the time evolution of the part of the core that has been colored in this way. The vortex ring moves normally to the plane of the photo graph in such a way that we see its front view. At the bo tom of the photographs the edge of the generator tube is visible. The portion *aa'* of the ring rotation axis that lie in the core is visible in Fig. 2, thanks to the air bubbles that have collected on it.

Leaving aside effects associated with the flows that exist along the core, it follows from examination of the

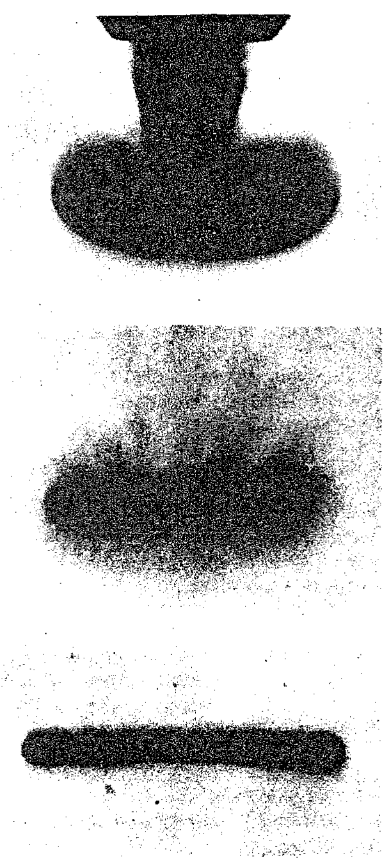

FIG. 1.

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