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POISSON-WIENER EXPANSION IN NON-LINEAR STOCHASTIC SYSTEMS

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Христо И. Христов. **Разложение Пуассона—Винера для нелинейных стохастических систем.** Рассматривается функциональное разложение Винера, в котором применяется процесс Пуассона в качестве базисной функции. Предлагается метод построения высших многоточечных моментов, что расширяет область применения метода Пуассона — Винера и на нелинейные стохастические уравнения. Основные результаты сформулированы также и при помощи дельта-коррелированного процесса — производной Пуассонового процесса, что существенно для приложений. Члены разложения интерпретированы физически, и техника применения метода демонстрирована на примере Бюргерсовой турбулентности.

Christo I. Christov. **Poisson-Wiener Expansion in Non-linear Stochastic Systems.** A kind of Wiener functional development of a non-linear functional with respect to the Poisson process as a basis function, so-called "Poisson — Wiener expansion" is considered. A method for handling the higher-order momenta of the multivariate Charlier polynomials of the Poisson process is proposed and formulae expressing the third momenta are derived. As a result, the region of application of the Poisson — Wiener method is expanded to non-linear stochastic systems and equations. A symbolic calculus based on the delta-correlated process (a derivative of the Poisson process called "perfectly white noise") is developed. An application to Burgers turbulence is outlined obtaining a first-order asymptotic solution and the physical meaning of the terms in Poisson — Wiener expansion is specified.

During the last several decades, the stochastic solutions of unstable non-linear systems as well as the solutions of systems with random coefficients have been in the focus of scientific attention due to their outstanding importance in a number of applications, such as turbulence, random noises, composite materials, multiphase flows, etc. After Keller and Friedmann [1] introduced the idea of infinite cascade system of equations, the main flow of papers on turbulence dealt with seeking possibilities to close this infinite system. Apart from semiempirical theories the most famous was Millionshtchikov's [2] quasi-normal assumption, so-called "zero-fourth-cumulant approach", until Y. Ogura [3] showed that this assumption led to negative energy spectrum in certain cases.

A completely different way was outlined by Wiener in 1930 (see [4]) applying to random functions the Volterra idea of representing any continuous functional in series of integer-order funtionals of certain given basis function. As a basis function he chose the process of Brownian motion. Cameron and Martin [5] built the rigorous mathematical basis of this method and coined the name "Wiener—Hermite expansion".

Wiener—Hermite method was applied to Burges' turbulence by Siegel et al. [6]. In succeeding paper these authors attained a number of further results, but it turned out that the time evolution of the kernels was unstable and needed a renormalization [7]. Now, it can be said that Wiener—Hermite expansion is faced with some difficulties in the case of non-linear systems because of the nature of its basis function — the Brownian-motion process. Evidently, the solutions of non-linear systems are well apart of Gaussianity which spoils the convergence of Wiener—Hermite series.

Meantime, there appeared a couple of papers in which the basis function was other than the Brownian-motion process. In particular, H. Ogura [8] introduced the Poisson process as a basis function in the Wiener development and translated most of Wiener's ideas to the case showing that the related orthogonal polynomials are the Charlier polynomials. After him this expansion was named "Poisson—Wiener expansion" The privileged position of the Poisson process in modelling the non-linear stochastic processes has been only recently shown by the author in [9].

The present paper sets forth to expose some further properties of the Charlier polynomials of Poisson process and to outline a way of application of Poisson—Wiener series displaying their unmatched performance in non-linear systems.

1. POISSON PROCESS AND DELTA-CORRELATED PROCESS

The following singular random process

$$f(t)=\sum_{-\infty\leqq\tau_k\leqq t}\delta(t-\tau_k),\quad -\infty\leqq t\leqq+\infty, \tag{1.1}$$

where the random points τ_k occur in accordance with the Poisson probability law is a particular case of the so-called shot-effect random process. The probability to find x impulses on the interval (s, t) is

$$P_n=\frac{[\gamma(t-s)]^x}{x!}\,e^{-\gamma(-s)}, \tag{1.2}$$

where γ is the mean number of points occurring per unit length. In addition, the probability to find certain number of points on a given interval is independent of the number of impulses which occur outside the considered interval (see for details [10]).

It is well known that the cumulants of a random variable x which is distributed according to the Poisson law (1.2) are given by

$$<x^n>_{\text{cuml}}=\gamma(t-s)\quad \text{for } n=1,\ 2,\dots\ . \tag{1.3}$$

Here $<\ \cdot\ >_{\text{cuml}}$ denotes the cumulant of the related average value. Respectively, the momenta of this variable are (see [10])

$$(1.4) \qquad \langle x\rangle = \langle x\rangle_{\text{cuml}} = \gamma(t-s),$$
$$\langle x^2\rangle = \gamma(t-s)+[\gamma(t-s)]^2,$$
$$\langle x^3\rangle = \gamma(t-s)+3[\gamma(t-s)]^2+[\gamma(t-s)]^3, \dots .$$
$$\vdots$$

Though $f(t)$ is a highly improper function one can formally write the following Lebesque integral

$$(1.5) \qquad D(t) = \int_{-\infty}^{t} f(s)\,ds.$$

The newly-defined process $D(t)$ consists of random steps with unit height Since the probability of occurring x steps in the interval (s, t) is given by (1.2) then

$$(1.6) \qquad \langle D(t)-D(s)\rangle_{\text{cuml}} = \langle D(t)-D(s)\rangle = \gamma(t-s)$$

because the height of each step is equal to unity. Let now $s=t-dt$. Then (1.6) readily yields

$$(1.7) \qquad \langle dD(t)\rangle = \gamma dt.$$

On the other hand, eqs. (1.3) give

$$(1.8) \qquad \langle [dD(t)]^n\rangle_{\text{cuml}} = \gamma dt, \quad n=1,\ 2,\dots .$$

As the numbers of points falling into disjoined intervals are stochastically independent the eqs. (1.8) transform to

$$(1.9) \qquad \langle dD(t)\rangle_{\text{cuml}} = \gamma\, dt,$$
$$\langle [dD(t_1)],\ [dD(t_2)]\rangle_{\text{cuml}} = \gamma\delta_{12}dt_1,$$
$$\langle [dD(t_1)], [dD(t_2)],\ \ [dD(t_3)]\rangle_{\text{cuml}} = \gamma\delta_{123}dt_1,$$

where $\delta_{12\dots n}$ is a Kronecer delta of n-th order.

On the other hand the differentials of $D(t)$ are easily expressed by means of function $f(t)$, i. e.

$$dD(t) \overset{\text{def}}{=} f(t)dt.$$

Then

$$\langle dD(t)\rangle_{\text{cuml}} = \langle f(t)\rangle_{\text{cuml}}\, dt$$

and therefore

$$\langle f(t)\rangle_{\text{cuml}} = \gamma.$$

In a similar manner it is derived that

$$\gamma\delta_{12}dt_1 = \langle [dD(t_1)],\ [dD(t_2)]\rangle_{\text{cuml}} = \langle f(t_1),\ f(t_2)\rangle_{\text{cuml}}\, dt_1 dt_2,$$

which yields

$$\langle f(t_1),\ f(t_2)\rangle_{\text{cuml}} = \gamma\frac{\delta_{12}}{dt_2} = \gamma\delta(t_1-t_2),$$

where $\delta(t_1-t_2)$ is the Dirac delta function. Let's denote

$$\Delta(t_i,\ t_j)=\delta(t_i-t_j), \tag{1.10}$$

$$\cdots\cdots$$

$$\Delta(t_1, \ldots, t_n)=\delta(t_1-t_2)\ldots\delta(t_1-t_n),$$

$$\cdots\cdots$$

Then

$$<f(t)>_{\text{cuml}} = \gamma, \tag{1.11}$$

$$<f(t_1),\ f(t_2)>_{\text{cuml}} = \gamma\Delta(t_1,\ t_2),$$

$$\vdots$$

$$<f(t_1),\ldots,\ f(t_n)>_{\text{cuml}} = \gamma\Delta(t_1,\ t_2,\ldots,\ t_n),$$

$$\cdots\cdots$$

The last equations explain the name "delta-correlated" for the process $f(t)$. Since the multivariate cumulants (the higher-order correlation functions, according to [10]) are delta functions, then the higher order multispectra of $f(t)$ are constants. This enables one to name $f(t)$ "perfectly white noise", as it was done in [11].

2. CHARLIER POLYNOMIALS

As it has been mentioned above, Cameron and Martin [5] developed Wiener's ideas on a more rigorous ground. They proved that the needed polynomials in Wiener expansion are the multivariate Hermite polynomials. It was related to Gaussian probability-distribution function of the Wiener process (Brownian-motion process) employed there. The success of the Wiener — Hermite expansion was so significant (for bibliography see [12]) that it took more than 20 years to come to employing other basis functions rather than the Wiener process. It was H. Ogura [8] who pointed out that the general theorem of [5] can be also applied to a development based on the Poisson random process. He found out that the related polynomials are the multivariate Charlier polynomials and coined the name Poisson—Wiener expansion.

A Charlier polynomial of n-th degree associated with the Poisson distribution with mean γ is defined [13] by

$$p_n(x\,;\ \gamma)=j(x\,;\ \gamma)^{-1}(-\Delta^n)\,j(x-n\,;\ \gamma), \tag{2.1}$$

where Δ indicates the difference operator $\Delta f(x)=f(x+1)-f(x)$. Using the displacement operator $Sf(x)=f(x-1)$, a recurrence formula

$$p_n(x\,;\ \gamma)=\left[\frac{x}{\gamma}\,S-1\right]p_{n-1}(x\,;\ \gamma) \tag{2.2}$$

follows from (2.1). Being reminded that $p_0(x,\ \gamma)=1$, the repeated use of (2.2) gives a representation for p_n:

$$p_n(x\,;\ \gamma)=\left[\frac{x}{\gamma}\,S-1\right]^n.\,1. \tag{2.3}$$

The explicit representation for p_n is given by the binomial expansion of the right-hand side of (2.3):

$$p_n(x\,;\,\gamma)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{r}\left(\frac{xS}{\gamma}\right)^{i}.\,1=\sum_{i=0}^{n}(-1)^{n-1}\binom{n}{i}\frac{x^{(i)}}{\gamma^{i}}, \tag{2.4}$$

where $x^{(i)}=x(x-1)\ldots(x-i+1)=(xS)^{i}.\,1$.

The first few polynomials are

$$\begin{aligned}
p_0(x\,;\,\gamma)&=1,\\
p_1(x\,;\,\gamma)&=\frac{x}{\gamma}-1,\\
p_2(x\,;\,\gamma)&=\frac{x(x-1)}{\gamma^2}-2\frac{x}{\gamma}+1,\\
p_3(x\,;\,\gamma)&=\frac{x(x-1)(x-2)}{\gamma^3}-3\frac{x(x-1)}{\gamma^2}+3\frac{x}{\gamma}-1,
\end{aligned} \tag{2.5}$$

etc. The orthogonality relation

$$\sum_{x=0}^{\infty}p_n(x\,;\,\gamma)p_m(x\,;\,\gamma)j(x\,;\,\gamma)=\langle p_n(x\,;\,\gamma)p_m(x\,;\,\gamma)\rangle=\varepsilon_{nm}\gamma^{-n}n\,! \tag{2.6}$$

can be proved using summation by parts [14]. Further, it is easily shown that

$$\sum_{x=0}^{\infty}p_1(x_i\,;\,\gamma)p_1(x_j\,;\,\gamma)p_1(x_k\,;\,\gamma)j(x;\,\gamma)=\gamma^{-2}\Delta(x_i,\,x_j,\,x_k). \tag{2.7}$$

Although it is not considered in the literature, the last property of Charlier polynomials is of outstanding importance, playing a decisive role in non-linear systems. Eq. (2.7) enables one to couple the equations for the third momenta on the "third" level with no need to attract fourth cumulants or fourth momenta.

In the same way can be derived

$$\begin{aligned}
\langle p_1(x_1\,;\,\gamma)\,p_1(x_2\,;\,\gamma)\,p_1(x_3\,;\,\gamma)\,p_1(x_4\,;\,\gamma)\rangle&=\gamma^{-3}\Delta(x_1,x_2,x_3,x_4)\\
&+\gamma^{-2}[\Delta(x_1,\,x_2)\Delta(x_3,\,x_4)+\Delta(x_1,\,x_3)\Delta(x_2,\,x_4)+\Delta(x_1,\,x_4)\Delta(x_2,\,x_3)].
\end{aligned} \tag{2.8}$$

It is important to stress here the parallel between the Wiener—Hermite expansion and Charlier polynomials. Namely, when $\gamma\gg1$, i. e. when a lot of points occur in unit interval, then the first term in (2.8) becomes less important than the second one and, in fact, (2.8) approaches the respective formula for the Hermite polynomials of the Brownian motion process. This means that the Poisson—Wiener development is in some sense more general than the Wiener—Hermite one and the latter is a limiting case of the former. The converse is not true and that is why certain processes which are close to the Poisson process cannot be successfully approximated by Wiener—Hermite series. In fact the higher order terms (kernels) tend to infinity in order to satisfy non-linearity or to attain Poisson shape. At the time the Poisson—Wiener series contains the required order of the respective terms by definition (see eq. (2.8)).

3. MULTIVARIATE CHARLIER POLYNOMIALS

The definition of Charlier polynomials can be generalized for many variables in a similar manner as it was done for the Hermite polynomials of many variables [10]. Introducing the Poisson distribution of s variables of means γ_s

$$j(x_1,\ldots, x_s;\ \gamma_1,\ldots, \gamma_s)=\prod_{i=1}^{s} e^{-\gamma_i}\frac{\gamma_i^{x_i}}{x_i\,!} \tag{3.1}$$

one can define (see [8])

$$\begin{aligned} &p^{(n)}(x_{i_1},\ldots, x_{i_n};\ \gamma_{i_1},\ldots, \gamma_{i_n})=\\ &=j(x_1,\ldots, x_s;\ \gamma_1,\ldots, \gamma_n)^{-1}\prod(-\Delta_{i_\nu}S_{i_\nu})\,j(x_1,\ldots, x_s;\ \gamma_1,\ldots, \gamma_s). \end{aligned} \tag{3.2}$$

An equivalent formulation is the following

$$\begin{aligned} &p^{(n)}(x_{i_1},\ldots, x_{i_n};\ \gamma_{i_1},\ldots, \gamma_{i_n})\\ &=j(x_1,\ldots, x_s;\ \gamma_1,\ldots, \gamma_s)^{-1}\prod(S_{i_\nu}-1)\,j(x_1,\ldots, x_s;\ \gamma_1,\ldots, \gamma_s), \end{aligned} \tag{3.3}$$

where the operators Δ_i or S_i are applied to variable x_i. For the sake of simplicity the parameters $\gamma_1\ldots, \gamma_s$ are often suppressed in j and $p^{(n)}$. It should be mentioned that s, the number of variables in (3.1) can be arbitrarily large.

Since the operators S_i commute, $p^{(n)}$ is symmetric with respect to its n variables. The recurrence formula

$$p^{(n)}(x_{i_1},\ldots, x_{i_n})=\left[\frac{x_{i_n}}{\gamma_{i_n}}S_{i_n}-1\right]p^{(n-1)}(x_{i_1},\ldots, x_{i_{n-1}}) \tag{3.4}$$

is derived in the same way as for the polynomials of a single variable. Respectively,

$$p^{(n)}(x_{i_1},\ldots, x_{i_n};\ \gamma_{i_1},\ldots, \gamma_{i_n})=\prod_{\nu=1}^{n}\left[\frac{x_{i_\nu}}{\gamma_{i_\nu}}S_{i_\nu}-1\right].1. \tag{3.5}$$

The explicit representation reads

$$\begin{aligned} &p^{(n)}(x_{i_1},\ldots, x_{i_n};\ \gamma_{i_1},\ldots, \gamma_{i_n})\\ &=\frac{x_{i_1}(x_{i_2}-\delta_{i_1i_2})\ldots(x_{i_n}-\delta_{i_1i_n}-\delta_{i_2i_n}-\ldots-\delta_{i_{n-1}i_n})}{\gamma_{i_1}\ldots\gamma_{i_n}}\\ &-\sum_{(n,n-1)}\frac{x_{j_1}(x_{j_2}-\delta_{j_1j_2})\ldots(x_{j_{n-1}}-\delta_{j_1j_{n-1}}-\ldots-\delta_{j_{n-2}j_{n-1}})}{\gamma_{j_1}\gamma_{j_2}\ldots\gamma_{j_n}} \end{aligned}$$

$$+ \ . \ . \ . \ +$$

$$+(-1)^{n-2}\sum_{(n,2)}\frac{x_{j_1}(x_{j_2}-\delta_{j_1 j_2})}{\gamma_{j_1}\gamma_{j_2}}+(-1)^{n-1}\sum_{(n,1)}\frac{x_{i_\nu}}{\gamma_{i_\nu}}+(-1)^n.$$

This expression does not show the symmetry of $p^{(n)}$ with respect to its n variables. Being reminded of certain obvious properties of Kronecer deltas, however, one can easily resurrect the symmetry. The first few polynomials are

(3.7)
$$p^{(0)}(x;\ \gamma)=1,$$

$$p^{(1)}(x;\ \gamma)=\frac{x}{\gamma}-1,$$

$$p^{(2)}(x_i,\ x_j;\ \gamma_i,\ \gamma_j)=\frac{x_i(x_j-\delta_{ij})}{\gamma_i\gamma_j}-\frac{x_i}{\gamma_i}-\frac{x_j}{\gamma_j}+1,$$

$$p^{(3)}(x_i,\ x_j,\ x_k;\ \gamma_i,\ \gamma_j,\ \gamma_k)=\frac{x_i(x_j-\delta_{ij})(x_k-\delta_{ik}-\delta_{jk})}{\gamma_i\gamma_j\gamma_k}$$

$$-\left[\frac{x_i(x_j-\delta_{ij})}{\gamma_i\gamma_j}+\frac{x_i(x_k-\delta_{jk})}{\gamma_j\gamma_k}+\frac{x_k(x_i-\delta_{ki})}{\gamma_k\gamma_i}\right]$$

$$+\left(\frac{x_i}{\gamma_i}+\frac{x_j}{\gamma_j}+\frac{x_k}{\gamma_k}\right)-1, \text{ etc.}$$

The most important feature of a multivariate Charlier polynomial is, perhaps, that it can be written as a product of several Charlier polynomials of a single variable, e. g.,

(3.8)
$$p^{(3)}(x_i,\ x_j,\ x_k)=\begin{cases} p_3(x_i) & ,\ i=j=k; \\ p_2(x_i)\ p_1(x_k) & ,\ i=j,\ i\neq k; \\ p_1(x_i)\ p_1(x_j)p_1(x_k), & i\neq j,\ j\neq k,\ k\neq i. \end{cases}$$

4. AVERAGE CHARACTERISTICS OF MULTIVARIATE CHARLIER POLYNOMIALS

The following orthogonality relation holds for $p^{(n)}$:

(4.1)
$$< p^{(n)}(x_{i_1},\ldots,x_{i_n})\ \gamma_{i_1},\ldots,\gamma_{i_n};\ p^{(m)}(x_{j_1},\ldots,x_{j_m};\ \gamma_{j_1},\ldots,\gamma_{j_m}) >$$

$$=\frac{1}{\gamma_{j_1}\cdots\gamma_{j_n}}\ \delta_{nm}\,\delta^n_{ij},$$

where brackets denote the average with respect to the distribution (3.1) while δ^n_{ij} equals the sum of all distinct products of n Kronecer deltas of the form $\delta_{i_\nu j_\mu}$, $i=(i_1,\ldots,i_m)$, $j=(j_1,\ldots,j_n)$, all i_ν and j_μ occurring only once in each product. Therefore δ^n_{ij} contains $n!$ terms. For example

$$(4.2)\qquad \delta^2_{ij}=\delta_{i_1j_1}\delta_{i_2j_2}+\delta_{i_1j_2}\delta_{i_2j_1}.$$

Relation (4.1) was proved by means of summation by parts in [8]. In the present paper it is derived from a different point of view which can be easily generalized for the products of more than two polynomials. Let $n\geqq m$ and let also $i_1, i_2, \ldots, i_n$ be all different. Then eq. (3.8) gives

$$(4.3)\qquad p^{(n)}(x_{i_1},\ldots, x_{i_n})=p^{(1)}(x_{i_1})\ldots p^{(1)}(x_{i_n}).$$

Being reminded that $<p^{(1)}(x_{i_\nu})>=0$ one finds that at least one of the first-order polynomials on the right-hand side of (4.3) remains unmatched by a responsible polynomial from $p^{(m)}$ if $m<n$, which reduces the average (4.1) to zero. The only case when this average deviates from zero is when $m=n$. Making use of (2.6), we derive:

$$(4.4)\qquad <p^{(n)}p^{(n)}>=\sum_{(n,n)}\prod_{\nu=1}^{n}<p^{(1)}(x_{i_\nu})p^{(1)}(x_{j_\mu})>=\frac{\delta^n_{ij}}{\gamma_{i_1}\ldots\gamma_{i_n}}.$$

Here all distinct combinations are accounted for in which i_ν and j_μ occur only once in each product. It is obvious that (4.1) and (4.2) are essentially the same.

The most important property of Charlier polynomials which makes them differ significantly from the Hermite set is the special type of their third momenta (see (2.7)). The latter is characteristic of the multivariate Charlier polynomials, too. Let's denote

$$(4.5)\qquad \Phi^{mnl}=<p^{(n)}(x_{i_1},\ldots, x_{i_n})\, p^{(m)}(x_{j_1},\ldots, x_{j_m})\, p^{(l)}(x_{k_1},\ldots, x_{k_l})>.$$

Without losing any generality only the case $n\leqq m\leqq l$ can be considered. In addition, it is required that $n+m\geqq l$, since the order of the product $p^{(n)}p^{(m)}$ is $n+m$, i. e. it is a sum of polynomials of order up to $n+m$. The latter means that (4.5) is reduced to a sum of binary products of polynomials of different orders: l and $l'\leqq m+n$ respectively, and the averages of these products are equal to zero when $l'<l$.

Once again k_λ should be al different, as well as all i_μ and all j_μ. The major difference between the case of second momenta (eq. 4.1)) and the case of third momenta is that in the latter one the nonzero averages can be obtained not only from the pair combinations of type (4.4) but also from the triple products of first-order polynomials. In order to assess the number of the latter, Φ^{mnl} is displayed in the form

$$(4.6)\qquad \Phi^{mnl}=<\underbrace{p^{(1)}(x_{k_1})\ldots p^{(1)}(x_{km})\,p^{(1)}(x_{km+1})\ldots p^{(1)}(x_{k_l})}_{l\ \text{terms}}$$

$$\times\underbrace{p^{(1)}(x_{j_1})\ldots p^{(1)}(x_{jm})}_{m\ \text{terms}}\ \underbrace{p^{(1)}(x_{i_1})\ldots p^{(1)}(x_{i_{l-m}})}_{l-m\ \text{terms}}$$

$$\times\underbrace{p^{(1)}(x_{i_{l-m+1}})\ldots p^{(1)}(x_{i_n})}_{n+m-l\ \text{terms}}>.$$

Now, the terms which are on the same vertical line are to be combined together and averaged, i. e.

$$\Phi_1=\sum \prod^{n+m-l} <p^{(1)}(x_{k_\lambda})\, p^{(1)}(x_{j_\mu})\, p^{(1)}(x_{i_\gamma})> \prod^{2l-n-m} <p^{(1)}(x_{k_{\lambda'}})\, p^{(1)}(x_{j_{\mu'}})>,$$

when $n+m=l$ one has a sum of double products, while for $m+n=2l$ the products are entirely comprised of triplets. The full number of such combinations (the number of components of the above sum) is

$$N_1=\frac{l!\,m!\,n!}{(l-m)!\,(l-n)!\,(m+n-l)!}. \tag{4.7}$$

Finally,

$$\Phi_1=\sum_{N_1}\frac{\delta^{n+m-l}_{ijk}\,\delta^{2l-n-m}_{j'k'}}{(\gamma_{k_1}\ldots\gamma_{k_{n+m-l}})^2\,(\gamma_{k_{n+m-l+1}},\ldots\gamma_l)}, \tag{4.8}$$

where

$$k=(k_1,\ldots,k_{m+n-l}),\quad k'=\begin{cases}(k_{m+n-l+1},\ldots k_l) & \text{for } l\geqq m+n-l+1,\\ \text{vanishes when} & l<m+n-l+1;\end{cases}$$

$$j=(j_1,\ldots,j_{m+n-l}),\quad j'=(j_{l-m-n+1},\ldots,j_m,\ i_1,\ldots,i_{l-m});$$

$$i=\begin{cases}(i_{n+m-l+1},\ldots,i_n) & \text{for } n\geqq n+m-l+1,\\ \text{vanishes} & \text{for } n<n+m-l+1.\end{cases}$$

In the same way can be treated the other contributors in Φ which are obtained by reducing the number of triplets in (4.6), e. g.,

$$\Phi_2=<\underbrace{p^{(1)}(x_{k_1})\ldots p^{(1)}(x_{k_{m-1}})\,p^{(1)}(x_{k_m})\ldots p^{(1)}(x_{k_l})p^{(1)}(x_{j_1})}_{l+1\ \text{terms}} \tag{4.9}$$

$$\times\underbrace{p^{(1)}(x_{j_2})\ldots p^{(1)}(x_{j_m})}_{m-1\ \text{terms}}\ \underbrace{p^{(1)}(x_{i_1})\ \ldots\ p^{(1)}(x_{i_{l-m+2}})}_{l-m+2\ \text{terms}}$$

$$\times\underbrace{p^{(1)}(x_{i_{l-m+3}})\ \ldots\ p^{(1)}(x_{i_n})}_{n+m-l-2\ \text{terms}}>.$$

Once again, it is easily proved that the number of terms in this sum is

$$N_2=\frac{(l+1)!\ n!\ (m-1)!}{(l+2-m)!\,(l-m+1)!\,(n+m-l-2)!}.$$

Respectively, eq. (4.9) is meaningful only when $m+n\geqq l+2$. For instance, when $l=m=n=2$ one has $N_2=6$ which is easily verified directly.

It seems there is no necessity to give all the explicit formulae of type (4.9). It is only necessary to assess here the powers of the products of gammas entering the related averages. For example, Φ_2 consists of the product of gammas of degree $(-m-n+1)$, Φ_3 — of $(-m-n+2)$, when it exists, etc.

The first few third momenta are

(4.10) $$<p^{(1)}(x_1)p^{(1)}(x_2)p^{(2)}(x_3, x_4)> = \frac{1}{\gamma_3\gamma_4}(\delta_{13}\delta_{24}+\delta_{14}\delta_{23}),$$

$$<p^{(1)}(x_1)p^{(2)}(x_2, x_3)p^{(2)}(x_4, x_5)>$$

$$= \frac{\delta_{421}\delta_{53}}{\gamma_4^2\gamma_5} + \frac{\delta_{431}\delta_{52}}{\gamma_4^2\gamma_5} + \frac{\delta_{521}\delta_{34}}{\gamma_5^2\gamma_4} + \frac{\delta_{531}\delta_{42}}{\gamma_5^2\gamma_4}$$

$$<p^{(2)}(x_1, x_2)p^{(2)}(x_3, x_4)p^{(2)}(x_5, x_6)>$$

$$= \frac{\delta_{531}\delta_{642}+\delta_{532}\delta_{641}+\delta_{541}\delta_{632}+\delta_{542}\delta_{631}}{\gamma_4^2\gamma_5^2}$$

$$+\frac{\delta_{14}\delta_{26}\delta_{35}+\delta_{15}\delta_{26}\delta_{34}+\delta_{14}\delta_{25}\delta_{36}+\delta_{15}\delta_{24}\delta_{36}+\delta_{16}\delta_{24}\delta_{35}+\delta_{16}\delta_{25}\delta_{34}}{\gamma_4\gamma_5\gamma_6}.$$

5. ORTHOGONAL FUNCTIONALS OF POISSON PROCESS

It is convenient to introduce the following set of polynomials

(5.1) $$c^{(n)}[x_1, .., x_n] = \gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_n}p^{(n)}(x_{i_1}, x_{i_2}, \dots, x_{i_n}; \gamma_{i_1}, \dots, \gamma_{i_n})$$

in the place of the original Charlier set [8].

If now the set of Poisson variables x_i are considered as increments of a given stationary in the strict sense Poisson process, i. e. $x_i = dD(t_i)$, then the number of points which occur (the means γ_i) is nothing but $\gamma_i = \gamma dt_i$, where γ is the mean of the Poisson proccess $D(t)$. Introducing this in (3.7), is obtained

(5.2) $$c^{(0)}[dD(t); \gamma dt] = 1,$$

$$c^{(1)}[dD(t_1); \gamma dt_1] = dD(t_1) - \gamma dt_1,$$

$$c^{(2)}[dD(t_1), dD(t_2); \gamma dt_1, \gamma dt_2] = dD(t_1)[dD(t_2) - \delta_{12}]$$

$$-\gamma[dD(t_1)dt_2 + dD(t_2)dt_1] + \gamma^2 dt_1 dt_2,$$

$$c^{(3)}[dD(t_1), dD(t_2), dD(t_3); \gamma dt_1, \gamma dt_2, \gamma dt_3]$$

$$= dD(t_1)[dD(t_2) - \delta_{12}][dD(t_3) - \delta_{13} - \delta_{23}]$$

$$-\gamma\{dD(t_1)[dD(t_2) - \delta_{12}]dt_3 + dD(t_2)[dD(t_3) - \delta_{23}]dt_1$$

$$+dD(t_3)[dD(t_1) - \delta_{31}]dt_2\} + \gamma^2\{dD(t_1)dt_2 dt_3$$

$$+dD(t_2)dt_1 dt_3 + dD(t_3)dt_1 dt_2\} - \gamma^3 dt_1 dt_2 dt_3, \text{ etc.}$$

Therefore, in place of eq. (4.1) one has

(5.3) $$<c^{(n)}[dD(t_{i_1}), \dots, dD(t_{i_n})]\, c^{(m)}[dD(t_{j_1}), \dots, dD(t_{j_m})]>$$

$$= \gamma^n \delta_{nm}\delta_{ij}^n dt_{i_1} dt_{i_2} \dots dt_{i_n}.$$

Further, all other properties are readily translated to the new set of polynomials

(5.4) $$< c^{(n)}[dD(t_{i_1}),\ldots, dD(t_{i_n})]c^{(m)}[dD(t_{j_1}),\ldots, dD(t_{j_m})]\, c^{(l)}[dD(t_{k_1}),\ldots, dD(t_{k_l})] >$$

$$=\gamma^l \sum_{N_1} \delta_{ijk}^{n+m-l}\,\delta_{j'k'}^{2l-n-m} dt_{k_1} dt_{k_2}\ldots dt_{k_l}$$

$$+\gamma^{l+1} \sum_{N_2} \delta_{i''j''k''}^{n+m-l-2}\,\delta_{j'''k'''}^{2l-n-m+3}\, dt_{k_1} dt_{k_2}\ldots dt_{k_l} dt_{j_1}+\ldots,$$

where $l \geqq m \geqq n$, $m+n \geqq l$ and the meaning of i, j, k, j', k', $N_{1,2}$ is the same as in (4.8). Respectively, the term of order γ^{l+1} exists only when $n+m \geqq l+2$; the term of order γ^{l+2} will exist only for $n+m \geqq l+4$; and so on.

The first few third momenta are

(5.5) $$< c^{(1)}[dD(t_1)]c^{(1)}[dD(t_2)\, c^{(2)}[dD(t_3),\ dD(t_4)] >$$
$$=\gamma^2[\delta_{13}\delta_{24}+\delta_{14}\delta_{23}]dt_3 dt_4,$$

(5.6) $$< c^{(1)}[dD(t_1)]\, c^{(2)}[dD(t_2),\ dD(t_3)]\, c^{(2)}[dD(t_4),\ dD(t_5)] >$$
$$=\gamma^2[\delta_{421}\delta_{53}+\delta_{431}\delta_{52}+\delta_{521}\delta_{43}+\delta_{531}\delta_{42}]dt_4\, dt_5,$$

(5.7) $$< c^{(2)}[dD(t_1),\ dD(t_2)]\, c^{(2)}[dD(t_3),\ dD(t_4)]\, c^{(2)}[dD(t_5),\ dD(t_6)] >$$
$$=\gamma^2\,[\delta_{642}\delta_{531}+\delta_{532}\delta_{641}+\delta_{541}\delta_{632}+\delta_{542}\delta_{631}]dt_5 dt_6$$
$$+\gamma^3[\delta_{15}\delta_{26}\delta_{34}+\delta_{14}\delta_{25}\delta_{36}+\delta_{15}\delta_{24}\delta_{36}+\delta_{16}\delta_{24}\delta_{35}$$
$$+\delta_{16}\delta_{25}\delta_{34}+\delta_{14}\delta_{26}\delta_{35}]dt_4 dt_5 dt_6.$$

Now, for any $\varphi \in L^2(R^n)$ one can define an n-tuple integral

(5.8) $$J_n(\varphi)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi(t_1,\ldots, t_n)\, c^{(n)}[dD(t_1),\ldots, dD(t_n)],$$

which is called "multiple Poisson—Wiener integral". Here $c^{(n)}[\,\cdot\,]$ is an analogue to the Wiener—Hermite functional $h^{(n)}[\,\cdot\,]$ and hence may be called the Poisson—Wiener functional. Some explicit forms of the multiple Poisson—Wiener integrals are:

(5.9) $$J_1(\varphi_1)=\int_{-\infty}^{\infty}\varphi_1(t)dD(t)-\gamma\int_{-\infty}^{\infty}\varphi_1(t)dt,$$

$$J_2(\varphi_2)=\iint_{-\infty}^{\infty}\varphi_2(t,\ t')dD(t)dD(t')-\int_{-\infty}^{\infty}\varphi_2(t,\ t)dD(t),$$

$$-2\gamma\int_{-\infty}^{\infty}\int\varphi_2(t,\ t')dt'dD(t)+\gamma^2\iint_{-\infty}^{\infty}\varphi_2(t,\ t')dtdt',\ \text{etc.}$$

The Poisson—Wiener functional spans an orthogonal subspace $L^{(n)}$ of $L^2[D]$. The Cameron—Martin theorem assures one that any non-linear functional of the Poisson process $F[D(\cdot)] \in L^2[D]$ can be developed in a series of multiple Poisson—Wiener integrals

$$(5.10) \qquad F[D(\cdot)] = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_n(t_1, \ldots, t_n)\, c^{(n)}[dD(t_1), \ldots, dD(t_n)].$$

This expansion can be used to represent any stochastic process assuming that φ_n are also functions of time t. For example, for $n=1$ one can consider the following stationary in the strict sense random process

$$I(t) = \int_{-\infty}^{\infty} \varphi(t-t')dD(t')$$

which is the well-known shot-noise process [10].

Generally, a stationary random process is expressible as

$$(5.11) \qquad I(t) = \varphi_0 + \int_{-\infty}^{\infty} \varphi_1(t-t')\, c^{(1)}[dD(t')]$$

$$+ \int\int_{-\infty}^{\infty} \varphi_2(t-t_1,\ t-t_2)\, c^{(2)}[dD(t_1),\ dD(t_2)] + \ldots$$

The stationarity applies here since $I(t)$ with t fixed is a non-linear functional of the Poisson process.

6. AN ALTERNATIVE FORMULATION

Although the derivative $f(t)$ of the Poisson process $D(t)$ is a highly improper function, in many cases it is more convenient to have the Poisson-Wiener expansion formulated in terms of that derivative. One can formally write

$$(6.1) \qquad dD(t) = f(t)dt.$$

On the other hand,

$$\int \varphi(t_1,\ t_2)\delta_{12}\, dt_1 = \int\int \varphi(t_1,\ t_2)\, \delta(t_1-t_2)\, dt_1 dt_2,$$

where $\delta(t_1-t_2)$ is the Dirac delta function. Being reminded of the adopted above notation

$$\Delta(t_{i_1},\ t_{i_2}, \ldots,\ t_{in}) = \delta(t_{i_1}-t_{i_2})\, \delta(t_{i_1}-t_{i_3}) \ldots \delta(t_{i_1}-t_{in}),$$

one can introduce the following set of functionals

$$(6.2) \qquad C^{(0)}[f(t)] = 1,$$

$$C^{(1)}[f(t_1)] = f(t_1) - \gamma,$$

$$C^{(2)}[f(t_1), f(t_2)] = f(t_1)[f(t_2) - \Delta(t_1, t_2)]$$

$$-\gamma[f(t_1) + f(t_2)] + \gamma^2$$

$$C^{(3)}[f(t_1), f(t_2), f(t_3)]$$

$$= f(t_1)[f(t_2) - \Delta(t_1, t_2)]\,[f(t_3) - \Delta(t_1, t_3) - \Delta(t_2, t_3)]$$

$$-\gamma\{f(t_1)[f(t_2) - \Delta(t_1, t_2)] + f(t_2)\,[f(t_3) - \Delta(t_2, t_3)]$$

$$+ f(t_3)[f(t_1) - \Delta(t_3, t_1)]\} + \gamma^2[f(t_1) + f(t_2) + f(t_3)] - \gamma^3$$

$$\cdots\cdots\cdots$$

$$C^{(n)}[f(t_{i_1}), \ldots, f(t_{i_n})]$$

$$= f(t_{i_1})\,[f(t_{i_2}) - \Delta(t_{i_1}, t_{i_2})] \ldots [f(t_{i_n}) - \Delta(t_{i_1}, t_{i_n}) - \ldots - \Delta(t_{i_{n-1}}, t_{i_n})]$$

$$-\gamma\left\{\sum_{(n,\,n-1)} f(t_{j_1})\,[f(t_{j_2}) - \Delta(t_{j_1}, t_{j_2})] \ldots [f(t_{j_{n-1}}) - \Delta(t_{j_1}, t_{j_{n-1}})\right.$$

$$\left. - \ldots - \Delta(t_{j_{n-2}}, t_{j_{n-1}})]\right\} + \ldots\ldots + \ldots$$

$$+\gamma^{n-2}(-1)^{n-2} \sum_{(n,\,2)} f(t_{j_1})\,[f(t_{j_2}) - \Delta[t_{j_1}, t_{j_2})]$$

$$+\gamma^{n-1}(-1)^{n-1} \sum_{\nu=1}^{n} f(t_{i_\nu}) + \gamma^n(-)^n,$$

which follows directly from (3.6) and (5.1).

The average properties of $c^{(n)}$ are easily translated to $C^{(n)}$, namely

(6.3) $$< C^{(n)}[f(t_{i_1}, \ldots, f(t_{i_n})]\, C^{(m)}[f(t_{j_1}), \ldots, f(t_{j_m})] >$$

$$= \gamma^n \delta_{nm} \Delta^n(i, j),$$

where $\Delta^n(i, j)$ equals the sum of all distinct products of n Dirac deltas of the form $\Delta(t_{i_\nu}, t_{j_\mu})$, $i = (i_1, \ldots, i_n)$ and $j = (j_1, \ldots, j_n)$ when all i_γ and j_μ occurring only once in each product. Hence $\Delta^n(i, j)$ contains $n!$ terms. For example

$$\Delta^2(i, j) = \Delta(t_{i_1}, t_{j_1})\,\Delta(t_{i_2}, t_{j_2}) + \Delta(t_{i_1}, t_{j_2})\,\Delta(t_{i_2}, t_{j_1})$$

$$= \delta(t_{i_1} - t_{j_1})\,\delta(t_{i_2} - t_{j_2}) + \delta(t_{i_1} - t_{j_2})\,\delta(t_{j_1} - t_{i_2}).$$

Turning to the third moments one has

(6.4) $$< C^{(n)}[f(t_{i_1}), \ldots, f(t_{i_n})]\, C^{(n)}[f(t_{j_1}), \ldots, f(t_{j_m})] C^{(l)}[f(t_{k_1}), \ldots, f(t_{k_l})] >$$

$$= \gamma^l \sum \Delta^{m+n-l}(i, j, k) \Delta^{2l-n-m}(j', k')$$

$$+ \gamma^{l+1} \sum \Delta^{n+m-l-2}(i'', j'', k'')\, \Delta^{2l-n-m+3}(j''', k''') + \ldots,$$

where $l \geqq m \geqq n$, $m+n \geqq l$ and the meaning of i, j, k and j', k' is the same as in (5.4), as well as $i'', j'', k'', j''', k'''$ and so on. For example

$$<C^{(1)}[f(t_1)]\, C^{(1)}[f(t_2)]\, C^{(2)}[f(t_3) f(t_4)] \tag{6.5}$$

$$=\gamma^2[\delta(t_1-t_3)\delta(t_2-t_4)+\delta(t_1-t_4)\delta(t_2-t_3)],$$

$$<C^{(1)}[f(t_1)]C^{(2)}[f(t_2),\ f(t_3)]C^{(2)}[f(t_4),\ f(t_5)]> \tag{6.6}$$

$$=\gamma^2[\delta(t_4-t_2)\delta(t_4-t_1)\delta(t_5-t_3)+\delta(t_4-t_3)\delta(t_4-t_1)\delta(t_5-t_2)$$

$$+\delta(t_5-t_2)\delta(t_5-t_1)\delta(t_4-t_3)+\delta(t_5-t_3)\delta(t_5-t_1)\delta(t_4-t_2)],$$

$$<C^{(2)}[f(t_1),\ f(t_2)]\, C^{(2)}[f(t_3),\ f(t_4)]C^{(2)}[f(t_5),\ f(t_6)]> \tag{6.7}$$

$$=\gamma^2[\delta(t_6-t_4)\delta(t_6-t_2)\delta(t_5-t_3)\delta(t_5-t_1)$$

$$+\delta(t_6-t_4)\delta(t_6-t_1)\delta(t_5-t_3)\delta(t_5-t_2)$$

$$+\delta(t_6-t_3)\delta(t_6-t_2)\delta(t_5-t_4)\delta(t_5-t_1)$$

$$+\delta(t_6-t_3)\delta(t_6-t_1)\delta(t_5-t_4)\delta(t_5-t_2)]$$

$$+\gamma^3[\delta(t_1-t_4)\delta(t_2-t_6)\delta(t_3-t_5)+\delta(t_1-t_5)\delta(t_2-t_6)\delta(t_3-t_4)$$

$$+\delta(t_1-t_4)\delta(t_2-t_5)\delta(t_3-t_6)+\delta(t_1-t_5)\delta(t_2-t_4)\delta(t_3-t_6)$$

$$+\delta(t_1-t_6)\delta(t_2-t_4)\delta(t_3-t_5)+\delta(t_1-t_6)\delta(t_2-t_5)\delta(t_3-t_4)].$$

Once again the connection with the Hermite—Wiener functionals can be traced. Indeed, for $\gamma \gg 1$ the second term on the right-hand side of (6.7) prevalis on the first one. At the time the second term is identical with that of the expression for the Wiener—Hermite third momentum.

The expression for a stationary in strict sense random process can be translated into a new set of functionals, namely,

$$I(t)=K_0+\int_{-\infty}^{\infty} K_1(t-t')C^{(1)}[f(t_1)]dt_1 \tag{6.8}$$

$$+\int\int_{-\infty}^{\infty} K_2(t-t_1,\ t-t_2)C^{(2)}[f(t_1),\ f(t_2)]\, dt_1 dt_2$$

$$+\int\int\int_{-\infty}^{\infty} K_3(t-t_1,\ t-t_2,\ t-t_3)C^{(3)}[f(t_1),\ f(t_2),\ f(t_3)]dt_1 dt_2 dt_3$$

$$+\ \ldots$$

The latter is a direct analogue and a corollary from (5.11). Here it should be noted that (6.8) is not rigorous enough, but it significantly facilitates the handling of the proposed technique. It should be understood as a symbolic calculus, while the rigorous basis is in section 5.

7. ORTHOGONAL DEVELOPMENT OF NON-LINEAR SYSTEMS IN SERIES OF POISSON-WIENER FUNCTIONALS

As has been mentioned above, the present technique is developed to attack essentially non-linear systems whose stochasticity is provided by an instability or is due to random coefficients. For this purpose it is necessary to develop the ralated mathematical background. Let u and v be two homogeneous in the strict sense random functions of variable x. They may depend also on some other parameters, e. g. the time t. These functions can be developed into the following series

$$\varphi = K_0^{\varphi}(\alpha) + \int_{-8}^{\infty} K_1^{\varphi}(x-\xi;\ \alpha) C^{(1)}[f(\xi)] d\xi \tag{7.1}$$

$$+ \int\int_{-\infty}^{\infty} K_2^{\varphi}(x-\xi_1,\ x-\xi_2;\ \alpha) C^{(2)}[f(\xi_1),\ f(\xi_2)] d\xi_1 d\xi_2 + \ldots,$$

where φ is either u or v and α is the vector of parameters.

Let now u and v be two unknowns which are to be estimated from certain non-linear system. The idea of the proposed method is in multiplying the basic system by $C^{(0)}$, $C^{(1)}[f(0)]$, $C^{(2)}[f(0), f(z)]$ etc. and averaging in order to obtain a system for the kernels K_0, K_1, $K_2, \ldots$ For doing this one needs the following formulae:

$$< C^{(0)}\varphi > = K_0^{\varphi} \tag{7.2}$$

$$< G^{(1)}[f(0)]\varphi > = \int_{-\infty}^{\infty} K_1^{\varphi}(x-\xi;\ \alpha) < C^{(1)}[f(\xi)] C^{(1)}[f(0)] > d\xi$$

$$= \gamma \int_{-\infty}^{\infty} K_1^{\varphi}(x-\xi;\ \alpha)\delta(\xi) d\xi = \gamma K_1^{\varphi}(x;\ \alpha),$$

$$< C^{(2)}[f(0),\ f(z)]\varphi > = \gamma^2 [K_2^{\varphi}(x,\ x-z;\ \alpha) + K_2^{\varphi}(x-z,\ x;\ \alpha)]$$

$$= 2\gamma^2 K_2^{\varphi}(x,\ x-z;\ \alpha), \text{ etc.}$$

The last equality is true because of the obvious symmetry of the multivariable kernels $K_n(x_1, \ldots, x_n)$ with respect to their arguments which is a corollary from the symmetry of the multivariate Poisson — Wiener functionals.

In the same fashion are treated the derivatives of φ with respect to the parameters α,

$$< C^{(0)} \nabla_\alpha \varphi > = \nabla_\alpha K_0^{\varphi}(\alpha), \tag{7.3}$$

$$< C^{(1)}[f(0)] \nabla_\alpha \varphi > = \gamma \nabla_\alpha K_1^{\varphi}(x;\ \alpha),$$

$$< C^{(2)}[f(0),\ f(z)] \nabla_\alpha \varphi > = 2\gamma^2 \nabla_\alpha K_2^{\varphi}(x,\ x-z;\ \alpha), \text{ etc.}$$

The derivative of φ with respect to the random variable x is

$$\frac{\partial\varphi}{\partial x}=\int_{-\infty}^{\infty}\frac{\partial}{\partial\eta}K_1^{\varphi}(x-\xi;\ \alpha)C^{(1)}[f(\xi)]d\xi$$

$$+\int\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\eta_1}+\frac{\partial}{\partial\eta_2}\right]K_2^{\varphi}(x-\xi_1,\ x-\xi_2;\ \alpha)C^{(2)}[f(\xi_1),\ f(\xi_2)]d\xi_1 d\xi_2$$

$$+\ldots,$$

where η denotes the random argument of $K_1(\eta;\ \alpha)$ and η_1, η_2 denote the first and the second random argument of $K_2(\eta_1,\ \eta_2;\ \alpha)$, respectively, and so on. Then

$$<C^{(0)}\frac{\partial\varphi}{\partial x}>=0, \tag{7.4}$$

$$<C^{(1)}[f(0)]\frac{\partial\varphi}{\partial x}>=\gamma\left.\frac{\partial K_1^{\varphi}}{\partial\eta}\right|_{\eta=x}=\gamma\frac{\partial K_1^{\varphi}(x;\alpha)}{\partial x},$$

$$<C^{(2)}[f(0),\ f(z)]\frac{\partial\varphi}{\partial x}>=\gamma^2\left[\left(\frac{\partial K_2^{\varphi}}{\partial\eta_1}+\frac{\partial K_2^{\varphi}}{\partial\eta_2}\right)_{\substack{\eta_1=x\\ \eta_2=x-z}}+\left(\frac{\partial K_2^{\varphi}}{\partial\eta_1}+\frac{\partial K_2^{\varphi}}{\partial\eta_2}\right)_{\substack{\eta_1=x-z\\ \eta_2=x}}\right]$$

$$=2\gamma^2\frac{\partial}{\partial x}K_2^{\varphi}(x,\ x-z;\ \alpha),\ \text{etc.}$$

Proceeding further, for the second derivatives it is obtained that

$$<C^{(0)}\frac{\partial^2\varphi}{\partial x^2}>=0, \tag{7.5}$$

$$<C^{(1)}[f(0)]\frac{\partial^2\varphi}{\partial x^2}>=\gamma\frac{\partial^2 K_1^{\varphi}(x;\ \alpha)}{\partial x^2},$$

$$<C^{(2)}[f(0),\ f(z)]\ \frac{\partial^2\varphi}{\partial x^2}>2\gamma^2\frac{\partial^2}{\partial x^2}K_2^{\varphi}\ (x,\ x-z;\ \alpha)\ \text{etc.}$$

All the above formulae are essentially similar to those for Wiener—Hermite functionals (see [6]). The crucial difference occurs when non linear terms are averaged, namely,

$$<C^{(0)}uv>=K_0^u K_0^v+\gamma\int_{-\infty}^{\infty}K_1^u(\xi;\ \alpha)K_1^v(\xi;\ \alpha)d\xi \tag{7.6}$$

$$+2!\ \gamma^2\int\int_{-\infty}^{\infty}K_2^u(\xi_1,\ \xi_2;\ \alpha)\,K_2^v(\xi_1,\ \xi_2;\ \alpha)d\xi_1\,d\xi_2+\ldots$$

$$<C^{(1)}[f(0)]uv>=\gamma[K_0^u K_1^v(x;\ \alpha)+K_0^v K_1^u(x;\ \alpha)+K_1^u(x;\ \alpha)K_1^v(x;\ \alpha)]$$

$$+\gamma^2\int_{-\infty}^{\infty}[2K_1^u(\xi)K_2^v(\xi,\ x)+2K_1^v(\xi)K_2^u(\xi,\ x)+4K_2^u(\xi,\ x)K_2^v(\xi,\ x)]d\xi+\ldots$$

$$(7.6'') \qquad < C^{(2)}[f(0), f(z)]uv> =$$
$$= \gamma^2\{K_1^u(x)K_1^v(x-z) + K_1^v(x)K_1^u(x-z)$$
$$+2!\,[K_0^u K_2^v(x, x-z) + K_0^v K_2^u(x, x-z)]$$
$$+2!\,[K_1^u(x) + K_1^u(x-z)]K_2^v(x, x-z)$$
$$+2!\,[K_1^v(x) - K_1^u(x-z)]K_2^u(x, x-z)$$
$$+(2!)^2 K_2^u(x, x-z)K_2^v(x, x-z)\} + \gamma^3\{\ .\ .\ .\ \} + \ .\ .\ .$$
$$(7.6''') \qquad < C^{(3)}[f(0), f(z), f(y)]uv> = \gamma^3\{\ .\ .\ .\ \} + \ .\ .\ .$$

Formulae (7.6′) — (7.6‴) feature the most important property of the Poisson—Wiener expansion: the ability of representing the non-linear character of the system on each level (for each kernel). Specifically, the average $< C^{(n)}uv>$ consists of the product $K_n^u K_n^v$ along with the other terms. Completely different is the case of Wiener—Hermite expansion where the non-linearity affects the equations for the kernels only through the higher-order terms. In some sense, the Poisson—Wiener expansion leaves the system "nonlinearly closed".

The Poisson—Wiener expansion possesses a clear physical meaning. The zero-order term gives the average value. The first-order term is the response of a non-linear instable system to a disturbance when the evolution of the latter is not affected by the presence of other disturbances. This case is possible when the system is near the threshold of instability and the eddies which appear due to instability are stable themselves, i. e. a new pattern can develop only after the previous one is nearly decayed. This is also the case with dilute suspensions and composite materials. The second order term is responsible for the interaction of the evolving eddies when only pair-interaction is taken into account. The third order term accounts for the triple interaction, etc.

Here it is to be reminded that the first- order term in Wiener—Hermite expansion accounts for Gaussianity while the higher-order terms are responsible for the deviation from Gaussianity. The latter has no clear physical meaning being purely mathematical approximation.

The Poisson—Wiener expansion is most desirable when the concentration γ of the eddies is low. Then it is reasonable enough to cut off the series even after the first-order term. In addition, this is the most frequently encountered in Nature case.

8. CALCULATING THE MAIN STATISTICAL PROPERTIES ON THE BASIS OF POISSON — WIENER EXPANSION

Let the homogeneous in the strict sense random function u be expanded in series of Poisson—Wiener functionals

$$(8.1) \qquad u = K_0 + \int_{-\infty}^{\infty} K_1(x-\xi)C^{(1)}[f(\xi)]d\xi$$
$$+ \int\!\!\int_{-\infty}^{\infty} K_2(x-\xi_1, x-\xi_2)C^{(2)}[f(\xi_1), f(\xi_2)]\, d\xi_1 d\xi_2 + \ .\ .$$

Here parameters α are suppressed as insignificant.

The mean value of u is

(8.2) $$<u> = K_0.$$

The second-order two-point momentum of u is expressed by

(8.3) $$<u(x)u(x+z)> = K_0^2 + \gamma \int_{-\infty}^{\infty} K_1(\xi+z)K_1(\xi)d\xi$$

$$+2!\,\gamma^2 \int\int_{-\infty}^{\infty} K_2(\xi_1, \xi_2)K_2(z+\xi_1, z+\xi_2)d\xi_1 d\xi_2 + \ldots$$

$$+n!\,\gamma^n \int \ldots \int_{-\infty}^{\infty} K_n(\xi_1, \ldots, \xi_n)K_n(z+\xi_1, \ldots, z+\xi_n)d\xi_1 d\xi_2 \ldots d\xi_n + \ldots$$

Therefore the energy of fluctuations is

(8.4) $$e = <u^2> - <u>^2 = \gamma \int_{-\infty}^{\infty} K_1^2(\xi)d\xi_1 + 2!\,\gamma^2 \int\int_{-\infty}^{\infty} K_2^2(\xi_1, \xi_2)d\xi_1 d\xi_2 + \ldots$$

Respectively, the coefficient of correlation is

(8.5) $$Q(z) = \frac{<u(x+z)u(x)> - <u(x+z)><u(x)>}{e}$$

$$= \frac{\gamma \int_{-\infty}^{\infty} K_1(\xi_1+z)K_1(\xi)d\xi_1 + 2!\,\gamma^2 \int\int_{-\infty}^{\infty} K_2(\xi_1+z, \xi_2+z)K_2(\xi_1, \xi_2)d\xi_1 d\xi_2 + \ldots}{\gamma \int_{-\infty}^{\infty} K_1^2(\xi)d\xi + 2!\,\gamma^2 \int\int_{-\infty}^{\infty} K_2^2(\xi_1, \xi_2)d\xi_1\, d\xi_2 + \ldots}.$$

On the basis of the latter the normalized energy spectrum can be calculated if necessary.

The most interesting feature of Poisson—Wiener expansion is connected with the third momentum

(8.6) $$<u(x)u(x+y)u(x+y+z)> = K_0^3$$

$$+\gamma \int_{-\infty}^{\infty} \{K_0[K_1(z+\xi) + K_1(z+y+\xi) + K_1(y+\xi)]K_1(\xi) + K_1(\xi)K_1(\xi+z)K_1(\xi+y+z)\}d\xi$$

$$+\gamma^2 \int\int_{-\infty}^{\infty} \{[K_2(z+\xi_1, z+\xi_2) + K_2(z+y+\xi_1, z+y+\xi_2) + K_2(y+\xi_1, y+\xi_2)]K_0 K_2(\xi_1, \xi_2)$$

$$+2K_2(z+\xi_1, \xi_2)K_1(\xi_1)K_1(y+\xi_2)$$

$$+2K_2(y+\xi_1, y+z+\xi_2)K_1(\xi_1)K_1(\xi_2)$$

$$+2K_2(\xi_1, \xi_2)K_1(z+\xi_1)K_2(y+z+\xi_2)$$

$$+4K_2(z+\xi_1,\ z+\xi_2)K_2(\xi_1,\ \xi_2)K_1(y+z+\xi_1)$$

$$+4K_2(y+z+\xi_1,\ y+z+\xi_2)K_2(z+\xi_1,\ z+\xi_2)K_1(\xi_1)$$

$$+4K_2(y+z+\xi_1,\ y+z+\xi_2)K_2(\xi_1,\ \xi_2)K_1(z+\xi_1)$$

$$+(2!)^2K_2(\xi_1,\ \xi_2)K_2(\xi_1+z,\ \xi_2+z)K_2(\xi_1+y+z,\ \xi_2+y+z)\}d\xi_1 d\xi_2$$

$$+\gamma^3\{\ .\ .\ \ \}+\ .\ .\ .\ ,$$

which is fully defined only by the first-order kernel within the order of accuracy $o(\gamma)$.

9. APPLICATION TO BURGERS TURBULENCE

Burgers [15] introduced the equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2} \tag{9.1}$$

as a substitute for the full Navier—Stokes system when one-dimensional shock waves in slightly compressible viscous fluid are considered. Here u is velocity, ν — kinematic coefficient of viscosity. Burgers' equation retains the major features of the full system, namely the non-linearity and viscosity, though it is much simpler. There are a lot of differences, too. The most important one is that Eq. (9.1) is never unstable and the stochasticity can only be forced into it, i. e. by means of random initial condition or by random disturbing force. Both these cases have received a vast numerical exploration in recent years. In our opinion the random-initial-condition method is more suited for qualitative modelling of the turbulence leaving more freedom to the intrinsic forces for governing the evolution of the stochastic solution. For this reason here is chosen the initial-condition Burgers turbulence as a featuring example displaying the method.

Consider the following initial condition

$$u(x,\ t=0)=UF(x), \tag{9.2}$$

where U is a measure of the initial velocity and $F(x)$ is a non-dimensional random function of the spatial variable x. The physical meaning of (9.2) is a train of weak shock waves. Let γ be the number of such shocks per unit length. Then $L=\gamma^{-1}$ is the length scale factor of function $F(x)$, and therefore the most suited Poisson process to be employed in the Poisson—Wiener expansion is that with a mean γ. Hence the stochastic solution of (9.1) under the random initial condition (9.2) is to be sought in the form

$$u(x,\ t)=K_0(t)+\int_{-\infty}^{\infty}K_1(x-\xi;\ t)\mathbf{C}^{(1)}[f(\xi)]d\xi \tag{9.3}$$

$$+\int_{-\infty}^{\infty}\!\!\int K_2(x-\xi_1,\ x-\xi_2;\ t)\mathbf{C}^{(2)}[f(\xi_1),\ f(\xi_2)]d\xi_1 d\xi_2+\ .\ .\ .$$

Introducing (9.3) into (9.1) and engaging all the technics of section 7, one obtains the following set of equations for the kernels K_n:

$$\frac{\partial K_0}{\partial t}=0. \tag{9.4'}$$

Without losing any generality one can take $K_0(t=0)=0$, i. e. $<F(x)>=0$. The latter means that a field with a zero mean is taken as an initial condition. Thus $K_0\equiv 0$ which allows one not to disregard K_0 in the equations for the higher-order terms. Further, multiplying by $C^{(1)}[f(0)]$ and taking the average, the following is obtained

$$\gamma\left[\frac{\partial K_1}{\partial t}+K_1\frac{\partial K_1}{\partial x}-\nu\frac{\partial^2 K_1}{\partial x^2}\right]$$

$$=\gamma^2\int_{-\infty}^{\infty}\left\{2K_2(x,\xi)\frac{\partial K_1}{\partial \xi}+[2K_1(\xi)+4K_2(x,\xi)]\left[\frac{\partial K_2(x,\xi)}{\partial \xi}+\frac{\partial K_2(x,\xi)}{\partial x}\right]\right\}d\xi$$

$$=\gamma^2\left\{2K_2(x,\xi)K_1(\xi)\Big|_{-\infty}^{\infty}+2\frac{\partial}{\partial x}\int_{-\infty}^{\infty}[K_1(\xi)K_2(x,\xi)+K_2^2(x,\xi)]\,d\xi+2K_2^2(x,\xi)\Big|_{-\infty}^{\infty}\right\}.$$

Being reminded that $\lim\limits_{\xi\to\pm\infty}K_2(x,\xi)=\lim\limits_{\xi\to\pm\infty}K_1(\xi)=0$, one obtains

$$\gamma\left[\frac{\partial K_1}{\partial t}+K_1\frac{\partial K_1}{\partial x}-\nu\frac{\partial^2 K_1}{\partial x^2}\right]=2\gamma^2\frac{\partial}{\partial x}\int_{-\infty}^{\infty}[K_1(\xi)K_2(x,\xi)+K_2^2(x,\xi)]d\xi. \tag{9.4''}$$

In the same fashion after multiplying by $C^{(2)}[f(0),f(z)]$ is derived

$$\begin{aligned}\gamma^2\Big\{2\frac{\partial K_2(x,x-z;t)}{\partial t}&+\frac{\partial}{\partial x}[K_1(x;t)K_1(x-z;t)]\\&+2\frac{\partial}{\partial x}[K_1(x;t)+K_1(x-z;t)]K_2(x,x-z;t)\\&+2\frac{\partial}{\partial x}K_2^2(x,x-z;t)-2\nu\frac{\partial^2}{\partial x^2}K_2(x,x-z;t)\Big\}=O(\gamma^3).\end{aligned} \tag{9.4'''}$$

Although K_2 is a function of two spatial arguments, the equation (9.4''') is effectively one-dimensional because the variable z plays the role of a parameter (there are no derivatives with respect to z)

The above system (9.4') — (9.4''') consists of time derivatives of the kernels K_n and is still too complex to be treated analytically. At long times, however, the initial condition is forgotten because of the dissipative character of the system. This raises the expectation that some kind of similar solution is attained as $t\to\infty$. The influence of the initial condition is expressed only in the magnitude of γ (the number of Poisson points occurring per unit length). Suppose this value is significantly low at the initial moment of time. It is reasonable enough then to expect that γ will evolve with time remaining relatively small with respect to the characteristic length $L=\sqrt{\nu t}$. This yields the condition

(9.5) $$\gamma=\frac{\varepsilon}{\sqrt{\nu t}}, \text{ where } \varepsilon\ll 1.$$

Analysis of dimensions requires that the similar solution be sought in the form

(9.6) $$K_1(x;\ t)=UB_1(\chi), \text{ where } \chi=\frac{x}{\sqrt{\nu t}},$$

$$K_2(x,\ x-z;\ t)=UB_2(\chi,\ \chi-\zeta),\ \zeta=\frac{z}{\sqrt{\nu t}}.$$

Introducing this into (9.4) and acknowledging the symmetry of K_2 with respect to its two argumeants, we obtain

(9.7′) $$-\frac{1}{2}(\chi B_1'+B_1)+B_1B_1'-B_1''(\chi)=\varepsilon\frac{\partial}{\partial x}\int_{-\infty}^{\infty}[B_1(\chi-\zeta)+B_2(\chi,\chi-\zeta)]\,B_2(\chi,\ \chi-\zeta)\,d\zeta$$

(9.7″) $$\frac{1}{2}\left(\chi\frac{\partial B_2}{\partial\chi}+B_2\right)+\frac{\partial^2}{\partial x^2}B_2(\chi,\ \chi-\zeta)$$
$$=\frac{\partial}{\partial x}[B_1(\chi)B_2(\chi-\zeta)+(B_1(\chi)+B_1(\chi-\zeta))B_2(\chi,\ \chi-\zeta)]+o(\varepsilon).$$

The terms of order $o(\varepsilon)$ in (9.7″) are not specified because they are not needed in constructing a solution of order $O(\varepsilon^2)$ since B_2 is multiplied by γ^2 in all formulae for statistical characteristics (see previous section).

Restricting ourselves to the order $O(\varepsilon^2)$, the solution is to be sought as follows

(9.8) $$B_1(\chi)=B_{10}(\chi)+\varepsilon B_{11}(\chi)+\ .\ .\ .\ ,$$
$$B_2(\chi,\ \chi-\zeta)=B_{20}(\chi,\ \chi-\zeta)+\ .\ .\ .\ ,$$

transforming (9.7) to the following system

(9.9) $$-\frac{1}{2}(\chi B_{10}'+B_{10})+B_{10}B_{10}'=B_{10}''(\chi),$$

(9.10) $$+\frac{1}{2}\left(\chi\frac{\partial B_{20}}{\partial\chi}+B_{20}\right)+\frac{\partial^2}{\partial\chi^2}B_{20}(\chi,\ \chi-\zeta)$$
$$=\frac{\partial}{\partial\chi}[B_{10}(\chi)B_{10}(\chi-\zeta)+(B_{10}(\chi)+B_{10}(\chi-\zeta))B_{20}(\chi,\ \chi-\zeta)+B_{20}^2(\chi,\ \chi-\zeta)],$$

(9.11) $$\frac{\partial}{\partial\chi}[B_{10}(\chi)B_{11}(\chi)]=\frac{\partial}{\partial\chi}\int_{-\infty}^{\infty}[B_{10}(\chi-\zeta)+B_{20}(\chi,\ \chi-\zeta)]B_2(\chi,\ \chi-\zeta)\,d\zeta.$$

The first of these equations is splitting off from the rest of the system which is an outstanding feature of the proposed method. The second amazing property of (9.9) is that it is non-linear although it is only a first approximation! At the time the first approximation in the Wiener—Hermite method is presented by a linear equation. In some sense the proposed here method is "non-linearly coupled". Hence it can be expected that even the first-order solution possesses the main information about stochastic properties of the full solution.

Eq. (9.9)—(9.11) are to be solved under the requirement of $L^2(R^n)$ space. In the author's work [9] it was found out that such a solution of (9.9) was

$$B_{10} = -\frac{4\chi}{2+\chi^2} \tag{9.12}$$

which showed a fair agreement of the correlation function, energy spectrum and energy rate of decaying with the numerical experiments [16, 17]. The major significance of (9.12) is that on its basis can be computed all the statistical characteristics of u with order of accuracy $o(\varepsilon)$. If a higher order of approximation is required Eq. (9.10) and (9.11) are to be solved. This goes beyond the frame of the present work whose purpose is to develop the mathematical background under the Poisson—Wiener functional expansion.

10. CONCLUSION

A kind of Wiener functional development of a non-linear functional with respect to Poisson process as a basis function, the so-called "Poisson—Wiener expansion" is considered. A method for handling the higher-order momenta of the multivariate Charlier polynomials of the Poisson process is proposed and formulae expressing the third momenta are derived. As a result, the region of application of the Poisson—Wiener method is expanded to non-linear stochastic systems and equations. A symbolic calculus based on the delta-correlated process which is a derivative of the Poisson process (called "perfectly white moise") is developed.

An application to the initial-value Burgers turbulence is outlined, obtaining a first-order asymptotic solution, and the physical meaning of the terms in the Poisson—Wiener expansion is specified.

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