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The Theory of a New Surface Dilational Viscometer

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The interfaces which separate the phases in chemical operator units have recently attracted considerable attention because of their importance in determining the mechanical, chemical and mass-transfer behaviour of the reacting systems. Knowledge of interfacial properties is essential to the process of theoretical modelling and improvement of chemical reactors employing multiphase flows.

The first use of intrinsic surface characteristics in a mechanical model was probably Lord Kelvin's [1] discovery of the major role played by surface tension in damping surface ripples. Measurement of ripples continues to be used today as an experimental technique for surface tension. Boussinesq [2] later assigned a dilational surface viscosity to a surface in an attempt to resolve the disagreement between the calculated and measured velocity of a falling liquid drop in another resting liquid. Scriven [3] proposed a two-dimensional Newtonian fluid model to describe the rheological properties of a surface. His model contains two surface viscosities: the surface shear and dilational viscosity. The generalization of the surface model was extended even further to include elastic [4] and non-Newtonian effects [5]. (In most of the so-called non-Newtonian effects, however, the apparent lack of adherence to the Newtonian model was actually due to the dependence of surface tension on changes of the concentration of adsorbed surfactant (see discussion in [6])).

The measurement of surface shear viscosity has proved fairly simple, since in one-dimensional shear flow, the surface tension does not contribute to the stresses [7, 8]. Dilational flow, however, has proven to be more intractable. In this case the surface tension gradient acts in conjunction with the viscous dilational force and exerts a more significant influence on flow patterns. The estimation of the surface dilational viscosity can thus be completely compromised, especially when the surfactant is soluble so that the surface tension acts as a damper.

Unlike shear flows, there exists no simple one-dimensional steady-state bulk flow able to provide a dilation on the surface. Among the two-dimensional steady-state bulk flows are thin films [10] and falling jets [11]. Unfortunately, these are quasi-steady flows and are unstable under certain conditions. In addition their theoretical modelling is not yet completed, rendering them unusable as viscometric experiments. Therefore, attention has focused on one-dimensional, but unsteady flows, which are represented mainly by the flow associated with the propagation of longitudinal wave on a surface [12].

It seems important to devise a two-dimensional steady-state bulk flow to measure the dilational surface viscosity. The present paper deals with the theory of such flow. The proposed flow is stable, steady and easily handled experimentally.

I. Governing Equations for a Plane Surface

Consider the surface of a surfactant solution. The mass balance equation for the surfactant at the surface is

$$\frac{\partial \Gamma}{\partial t} + div(\vec{v}\Gamma) = j(c, \Gamma), \tag{1.1}$$

where $\vec{v}$ is the surface velocity, c the bulk concentration of the surfactant and j (c, Γ) is the rate of sorption process. The momentum balance equation can be written as [3]:

$$\varrho_s \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = \nabla\sigma + \varkappa\nabla^2 \vec{v} + \varkappa' \operatorname{grad} (\nabla \cdot \vec{v}) + F_{dr}, \tag{1.2}$$

where ϱ_s is the surface density, σ is the surface tension, $\varkappa'$ is the surface dilational viscosity and $\varkappa$ is the surface shear viscosity, and F_{dr} is the drag force exerted on the surface from the bulk. The term involving density in (1.2) can be neglected since $\varrho_s \approx 10^{-6}$ g/cm^2.

We consider the limiting case where sorption process in controlling and assume that the Langmuir adsorption law holds, as is suggested in [13]:

$$j(c, \Gamma) = A_1 c(c\Gamma_\infty - \Gamma) - A_2 \Gamma,$$

where A_1 is the adsorption rate constant, A_2 is the desorption rate constant and Γ_∞ is the surface concentration corresponding to an infinite value for the bulk concentration. If the chnge of Γ and c is small, one can linearize (1.1) as follows:

$$\frac{\delta \overline{\Gamma}}{\delta t} + \Gamma_0, \; div\vec{v} = A_1(\Gamma_\infty - \Gamma)\overline{c} - (A_1 c_0 + A_2)\overline{\Gamma}, \tag{1.3}$$

where $\overline{\Gamma} = \Gamma - \Gamma_0$ and $\overline{c} = c - c_0$. The term with the bulk concentration can be neglected only when

$$\frac{A_1(\Gamma_\infty - \Gamma_0)}{A_1 c_0 + A_2} \; \frac{c_0}{\Gamma_0} \ll 1.$$

The last condition is satisfied for sufficiently high bulk concentration c_0, when the surface concentration Γ_0 approaches Γ_∞. This is then the most useful case.

The surface tension is related to the surface concentration through the equation of state $\sigma = \sigma(\Gamma)$. Linearizing the equation of state gives:

$$(\sigma - \sigma_0) = (\Gamma - \Gamma_0)\frac{d\sigma}{d\Gamma}\bigg|_{\Gamma=\Gamma_0} = -\frac{\overline{\Gamma}}{\Gamma_0} E_0, \tag{1.4}$$

where $E_0=-\Gamma_0\frac{d\sigma}{d\Gamma_0}$ is called Gibb's elasticity. Making use of (1.3) and (1.4) one finds

$$(1.5)\qquad \frac{1}{A_1c_0+A_2}\,\frac{\partial\sigma}{\partial t}+\sigma=\frac{E_0}{A_1c_0+A_0}\,div\,\vec{v}.$$

This result asserts that, under the above assumptions the surface exhibits an apparent Maxwellian viscoelastic behaviour with a relaxtion time $t_m=1/(A_1c_0+A_2)$.

When the surface motion has only one nonzero velocity component v_x and none of the variables depends on coordinate y, then eq. (1.2) reduces to:

$$(\varkappa+\varkappa')\,\frac{d^2v_x}{dx^2}+F_{dr}+\frac{d\sigma}{dx}=0.$$

Eq. (1.5) qives

$$\sigma=\sigma_0+\frac{E_0}{A_1c_0+A_2}\,\frac{dv_x}{dx}$$

and the simple substitution in the above equality yields

$$(\varkappa+\varkappa'+\eta_{cd})\frac{d^2v_x}{dx^2}+F_{dr}=0,$$

where $\eta_{cd}=\frac{E_0}{A_1c_0+A_2}$ is usually called the "compositional" dilational viscosity It is convenient to introduce $\eta=\varkappa+\varkappa'+\eta_{cd}$ and to name it "net" dilational viscosity. Finally, the governing equation for one-dimensional steady surface motion can be rewritten in the form:

$$(1.6)\qquad \eta\,\frac{d^2v}{dx^2}+F_{dr}=0,$$

where the subscript x of the velocity component is omitted for the sake of brevity.

It is evident that neither of the viscosity coefficients plays an independent role. Rather, their sum, the "net" viscosity is used in the equation of motion. The present paper deals with that "net" coefficient.

II. Description of the Method and Coordinates

Consider a rectangular cavity filled with an incompressible Newtonian fluid. A cylinder with a radius a is submerged in the liquid with its axis parallel the y-axis and is set in motion with an angular velocity Ω, as shown in Fig. 1. It is assumed that the distance between the top of the cylinder and the free surface d is much less than the radius of the cylinder a,

$$d\ll a.$$

This assumption assures that the flow in the vicinity of point O will not be significantly affected by the wall.

The most convenient coordinates to describe the region between a flat surface and a circular cylinder are the bipolar coordinates α and β, which relate to the Cartesian coordinates x, z as follows:

$$(2.1)\qquad z=s\frac{sh\,\beta}{ch\,\beta-\cos\alpha}\quad\text{and}\quad x=s\,\frac{\sin\alpha}{ch\,\beta-\cos\alpha}.$$

A detailed description of the bipolar coordinates with application to slow viscous motions can be found in [15]. Bipolar coordinates are convenient for this case, because both the cylinder and the surface are coordinate lines with the surface represented by $\beta=0$ and the cylinder by $\beta=\varepsilon$. ε and s can be expressed in terms of a and d as follows:

$$\varepsilon = arc\ sh\left(\frac{s}{a}\right) \quad \text{and} \quad s=\sqrt{d(d+2a)}. \tag{2.2}$$

When $d \ll a$ eqs. (2.2) can be approximated as follows:

$$\varepsilon \approx \sqrt{\frac{2d}{a}} \ll 1 \quad \text{and} \quad s \approx \sqrt{2da}. \tag{2.3}$$

The region around point O, in which the disturbances due to the presence of walls are negligible, can be specified as $\frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}$. The value $\alpha=\pi$ corresponds to the point O itself, while the curves $\alpha=\frac{\pi}{2}$ and $\alpha=\frac{3\pi}{2}$ are the two halfs of the circle with a center at point O and radius s. It is obvious that $\frac{s}{a}=\sqrt{\frac{2d}{a}}=\varepsilon$, i. e. the prescribed region occupies a share proportional to the small parameter ε; in terms of variable x, it is the interval $-s \leq x \leq s$.

III. Fluid Flow between the Rotating Cylinder and the Surface for Which Velocity is Prescribed

The set of two-dimensional Navier-Stokes equations and the continuity equation can be rendered in terms of the non-dimensional stream function ψ and the vorticity ζ as functions of the bipolar coordinates α and β:

$$\frac{\partial^2 \zeta}{\partial \alpha^2} + \frac{\partial^2 \zeta}{\partial \beta^2} = Re\left(\frac{\partial}{\partial \beta^2}\frac{\partial \zeta}{\partial \alpha} - \frac{\partial}{\partial \alpha}\frac{\partial \xi}{\partial \beta}\right) \tag{3.1}$$

$$(ch\,\beta - cos\,\alpha)^2\left(\frac{\partial^2 \psi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \beta^2}\right) = -\zeta, \tag{3.2}$$

where $Re=\frac{\Omega a^2}{\nu}$ is the Reynolds number. The velocity components are related to the stream function ψ as follows:

$$v_\alpha = a\Omega(ch\,\beta - cos\,\alpha)\frac{\partial \psi}{\partial \beta}, \quad v_\beta = -a\Omega(ch\beta - cos\,\alpha)\frac{\partial \psi}{\partial \alpha}. \tag{3.3}$$

The presence of the small parameter ε in the system (3.1), (3.2) allows one to disregard some of the terms in these equations. The derivatives $\frac{\partial}{\partial \beta}$ are of order $O\left(\frac{1}{\varepsilon}\right)$ at the time when $\frac{\partial}{\partial \alpha}$ is of unit order, making latter term negligible in comparison with the former. Moreover, the left-hand side of (3.1) is of order $O\left(\frac{1}{\varepsilon^2}\right)$ while the right-hand side is of $O(Re)$, because the nondimensional ψ and ζ are of unit order. Assuming now that Reynolds number is also of unit order, the two equations (3.1), (3.2) can be reduced asymptotically to the following equation:

$$(1-cos\,\alpha)^2\frac{\partial^4 \psi}{\partial \beta^4} = Re\,.\,O(\varepsilon^2), \tag{3.4}$$

where $\bar{\beta}=\beta/\varepsilon$. In addition, $ch\,\beta\approx 1+O(\varepsilon^2)$ when $\beta\ll\varepsilon$. If $\varepsilon^2\ll 1$, or, in other words, if $2d\ll a$, then the right-hand side of (3.4) can be neglected. Practically speaking, it is reasonable to require that $d\leq 0.02a$, so that eq. (3.4) finally reduces to

$$\frac{\partial^4\psi}{\partial\bar{\beta}^4}=0. \tag{3.5}$$

The boundary conditions for this equation are:

$$(1-\cos a)\,\frac{\partial\psi}{\partial\bar{\beta}}=\bar{v}=\frac{v(a)}{a\Omega} \quad \text{and} \quad (1-\cos a)\,\frac{\partial\psi}{\partial a}=0 \text{ at } \beta=0, \tag{3.6}$$

where $v(a)$ is the as yet unknown surface velocity, and;

$$(1-\cos a)\,\frac{\partial\psi}{\partial\bar{\beta}}=1 \quad \text{and} \quad (1-\cos a)\,\frac{\partial\psi}{\partial a}=0 \text{ at } \bar{\beta}=1 \ (\beta=\varepsilon). \tag{3.7}$$

It can be shown that the solution of the boundary value problem (3.5)–(3.7) is

$$\psi=\frac{\beta^3}{\varepsilon^2}\left[\frac{1+\bar{v}(a)}{1-\cos a}-2M\right]-\frac{\beta^2}{\varepsilon}\left[\frac{1+2\bar{v}(a)}{1-\cos a}-3M\right]+\beta\,\frac{\bar{v}(a)}{1-\cos a}. \tag{3.8}$$

Here, M is an unknown number which represents the value of the nondimensional stream function at the cylinder. The value of ψ at the free surface is already taken $\psi=0$. The constant M appears in the solution because the region of integration is not single-connected. It is well known that in double-connected regions the equations of Navier-Stokes possess more than one solution. However, only one of the solutions is continuous, and M here is to be determined through the requirement that it be a single-valued solution for the pressure, i. e. that the pressure should be a periodic function of a. Considering the Navier-stokes equation for the a component of the velocity, and neglecting the terms of higher order one obtains:

$$(1-\cos a)\,\frac{\partial p}{\partial a}=\frac{\mu\Omega a}{s}(1-\cos a)^3\,\frac{\partial^3\psi}{\partial\bar{\beta}^3}. \tag{3.9}$$

When the pressure is a periodic function, then $0=\int_0^{2\pi}\frac{\partial p}{\partial a}\,da$, which yields

$$\int_0^{2\pi}[(1+\bar{v})(1-\cos a)^2-2M(1-\cos a)^3]\,da=0. \tag{3.10a}$$

The last relation leads to the following:

$$M=\frac{3}{10}+\frac{1}{10}\int_0^{2\pi}\bar{v}(a)(1-\cos a)^2 da. \tag{3.10b}$$

It is curious to note that in the case of an immobile surface, when $v(a)=0$ the total nondimensional flux is $M=0.3$. For the pure-water surface, when $\frac{\partial^2\psi}{\partial\bar{\beta}^2}=0$, one finds $M=0.6$, i. e. the flux is two times greater.

Now, the drag force exerted from the bulk on the surface can be calculated as follows:

$$F_{dr}=\mu \frac{\partial v_\alpha}{\partial \beta}\left(\frac{1-\cos \alpha}{s}\right)\bigg|_{\beta=0}=\frac{\mu \Omega a}{s}(1-\cos \alpha)^2 \frac{\partial^2 \psi}{\partial \beta^2}\bigg|_{\beta=0}.$$

Substituting here the expression for ψ yields:

$$(3.11)\quad F_{dr}=\frac{\mu \Omega a}{s}(1-\cos \alpha)^2\left[\frac{1+2\bar{v}}{1-\cos \alpha}-\frac{9}{10}-\frac{3}{10\pi}\int_0^{2\pi}\bar{v}(\alpha)(1-\cos \alpha)^2 d\alpha\right].$$

It is now convenient to represent the drag force in terms of the cartesian variable x. Indeed, at $\beta=0$ (the surface) the following connection between x, α holds:

$$\frac{x}{s}=\frac{\sin \alpha}{1-\cos \alpha},\quad \cos \alpha=\frac{x^2-s^2}{x^2+s^2} \quad \text{and} \quad dx=-s\frac{d\alpha}{1-\cos \alpha}.$$

Then, since $\varepsilon=s/a$, one obtains

$$(3.12)\quad F_{dr}=\mu a \frac{4s^2}{(x^2+s^2)^2}\left\{\frac{x^2+s^2}{2s^2}[a\Omega+2v(x)]-9/10\, a\Omega+\frac{3}{10\pi}\int_{-\infty}^{\infty}\frac{8s^5 y(x)dx}{(x^2+s^2)^3}\right\}.$$

IV. The Equation of Surface Motion and the Relation for Estimating the Surface Viscosity

Formula (3.12) can be substituted into the equation of surface motion (1.6), as follows:

$$(4.1)\quad \eta \frac{d^2v}{dx^2}+\mu a \frac{4s^2}{(x^2+s^2)^2}\left\{\frac{x^2+s^2}{2s^2}[a\Omega+2v(x)]-\frac{9}{10}a\Omega+\frac{3}{10\pi}\int_{-\infty}^{\infty}\frac{8s^5 y(x)}{(x^2+s^2)^3}dx\right\}=0.$$

It is convenient to rewrite (4.1) in nondimensional form,

$$(4.2)\quad \frac{\eta s^2}{\mu a}\frac{d^2\bar{v}}{dx^2}+\frac{4}{(\bar{x}^2+1)^2}\left[\frac{x^2+1}{2}(1+2\bar{v})-\frac{9}{10}+\frac{12}{5\pi}\int_{-\infty}^{\infty}\frac{\bar{v}\,d\bar{x}}{(\bar{x}^2+1)^3}\right]=0,$$

where $x=s\bar{x}$ and $v=a\Omega\bar{v}$.

Obviously, the region around point 0 is represented by $x \in [-1, 1]$. In addition, a nondimensional viscosity is defined as $\lambda=\frac{\eta s^2}{\mu a}$, which is the only parameter of eq. (4.2)! The latter can not be solved without boundary conditions; unfortunately, boundary conditions can be obtained only after matching to the "inner" solution in the asymptotic scheme adopted here. The "inner" solution is that in the region near $\alpha=0$ and $\alpha=2\pi$. The exact form of the "inner" solution can be specified only by means of calculations which are of the same level of complexity as the full problem itself, dispelling all the advantages of the simple solution of the previous section. Moreover, in order to estimate the quantity λ, one need not find the solution of (4.2); it is enough, rather, to measure the velocity v at certain points in the region around point 0 and to turn (4.2) into an equation for λ, rather than v. The only obstacle to this strategy is the integral term in (4.2), which requires a knowledge of

$wv(x)$ on the entire interval $(-\infty, \infty)$. After some obvious manipulations, hoever, eq. (4.2) yields :

$$\frac{\lambda}{4}\frac{d}{dx}(\bar{x}^2+1)^2\frac{d^2\bar{y}}{d\bar{x}^2}+\frac{d}{d\bar{x}}\left[\frac{(\bar{x}^2+1)(1+2\bar{v})}{2}\right]=0.$$

It is easy to show that

$$(4.3)\qquad \frac{\lambda}{4}\left[4\bar{x}(\bar{x}^2+1)\frac{d^2\bar{v}}{d\bar{x}^2}+(\bar{x}^2+1)^2\frac{d^3\bar{v}}{d\bar{x}^3}\right]+\bar{x}(1+2\bar{v})+(\bar{x}^2+1)\frac{l\bar{v}}{d\bar{x}}=0.$$

Since the velocity $\bar{v}(x)$ is an even function, one can represent it in a Taylor series only with even powers of x:

$$\bar{v}=\bar{v}(0)+\frac{\bar{x}^2 d^2\bar{v}}{2d\bar{x}^2}\bigg|_{\bar{x}=0}+\frac{\bar{x}^4}{4!}\frac{d^4\bar{v}}{d\bar{x}^4}\bigg|_{\bar{x}=0}+\cdots.$$

Then

$$\frac{d\bar{v}}{d\bar{x}}=\bar{x}\frac{d^2\bar{v}}{d\bar{x}^2}\bigg|_{x=0}+\frac{\bar{x}^3}{3!}\frac{d^4\bar{v}}{d\bar{x}^4}\bigg|_{\bar{x}=0}+\cdots$$

and

$$\frac{d^3\bar{v}}{d\bar{x}^3}=\bar{x}\frac{d^4\bar{v}}{d\bar{x}^4}\bigg|_{x=0}+\cdots.$$

As has been mentioned above, point O is represented by $\bar{x}=0$. An evaluation of (4.2) limited to the vicinity around point O can be accomplished using only the first terms of the above Taylor representations, yielding

$$\lambda\left[4\bar{x}(\bar{x}^2+1)\frac{d^2\bar{v}}{d\bar{x}^2}\bigg|_{\bar{x}=0}+\bar{x}(\bar{x}^2+1)^2\frac{d^4\bar{v}}{d\bar{x}^4}\bigg|_{\bar{x}=0}\right]=-\bar{x}(1+2\bar{v})-\bar{x}(\bar{x}^2+1)\frac{d^2\bar{v}}{d\bar{x}^2}.$$

It is evident that $\bar{x}$ can be cancelled from both sides of this equality, i. e.

$$\lambda\left[(\bar{x}^2+1)\frac{d^2\bar{v}}{d\bar{x}^2}\bigg|_{\bar{x}=0}+\frac{(\bar{x}^2+1)^2}{4}\bigg|_{\bar{x}=0}\right]=-(1+2\bar{v})-(\bar{x}^2+1)\frac{d^2\bar{v}}{d\bar{x}^2}\bigg|_{\bar{x}=0}.$$

Taking here the limit $x\to 0$, the following equation for λ is found:

$$(4.4)\qquad \lambda=-\frac{1+2\bar{v}(0)+\dfrac{d^2\bar{v}}{d\bar{x}^2}\bigg|_{\bar{x}=0}}{\dfrac{d^2\bar{v}}{d\bar{x}^2}\bigg|_{\bar{x}=0}+\dfrac{1}{4}\dfrac{d^4\bar{v}}{d\bar{x}^4}\bigg|_{\bar{x}=0}},$$

where $\bar{x}$ and $\bar{v}$ are the nondimensional surface coordinate and surface velocity respectively.

For the practical estimation of λ, it is necessary to measure the velocity at a minimum of three surface points, including the origin O, represented by

$$v_0=\bar{v}(0),\qquad v_1=\bar{v}(h)\quad\text{and}\quad v_2=\bar{v}(2h),$$

where h is a small value in comparison with unity. The symmetry of v adds the following relations:

$$v_{-1}=\bar{v}(-h)=\bar{v}(h)=v_1 \quad \text{and} \quad v_{-2}=\bar{v}(-2h)=\bar{v}(2h)=v_2.$$

So far, the derivatives of $\bar{v}$ employed in (4.4) can be expressed approximately as follows

$$\left.\frac{d^2\bar{v}}{d\bar{x}^2}\right|_{\bar{x}=0}=\frac{v_{-1}-2v_0+v_1}{h^2}+O(h^2)=\frac{2(v_1-v_0)}{h_2}+O(h^2)$$

and

$$\left.\frac{d^4\bar{v}}{d\bar{x}^4}\right|_{x=0}=\frac{v_{-2}-4v_{-1}+6v_0-4v_1+v_2}{h^4}+O(h^2)=\frac{2(v_2-4v_1+3v_0)}{h^4}+O(h^2).$$

Then (4.4) transforms to:

$$\lambda=\frac{1+2v_0+\dfrac{2(v_1-v_0)}{h^2}}{\dfrac{2(v_1-v_0)}{h^2}+\dfrac{v_2-4v_1+3v_0}{2h^4}}+O(h^2), \tag{4.5}$$

which allows an estimation of λ from the velocity measurements. If a greater accuracy is required, then further velocity measurements must be performed. For example, incorporating the values of $\bar{v}(3h)$ and $\bar{v}(4h)$ yields an accuracy of order $O(h^4)$ in formula (4.5).

V. Conclusions

In the present work, the viscous flow around a long cylinder of radius a immersed along distance d under a free surface has been investigated. An approximate solution of the Navier—Stokes equations with an arbitrary surface velocity as a boundary condition has been found when $d \ll a$. On the basis of this solution, the drag force exerted on the surface from the bulk has been calculated and thus a closed equation for the surface motion void of any bulk variables has been derived. The surface has been presumed to be a Newtonian two-dimensional liquid with intrinsic shear and dilational viscosity. Assuming also that there is adsorption-controlled mass interchange between the surface and the bulk, the surface tension has been shown to act as an additional resistance to the dilational strains. As a result, the equation of surface motion consists of only one parameter: the "net" dilational viscosity η. Though this equation has not been solved, due to the absence of boundary conditions in the adopted asymptotic scheme, it has been shown to be useful as a relation for the dilational-viscosity coefficient η when the surface velocity is thought of as a known function. A practical method for making this estimate has been outlined, based on an approximation of the equation for surface motion.

The proposed method for estimating the "net" coefficient of dilational viscosity

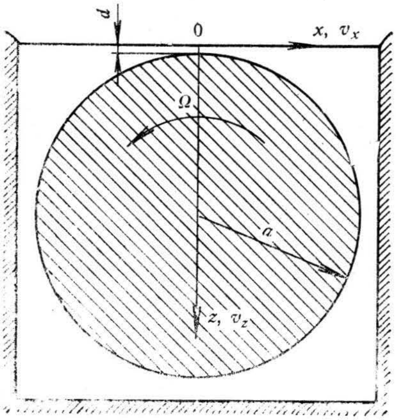

Fig. 1

can be used if the measures of the apparatus (fig. 1) obey the following limitations:

(i) The thickness d of the liquid layer between the cylinder and the surface is less than $0.02a$, where a is the cylinder radius.

(ii) The Reynolds number $\Omega a^2/\nu$ is of unit order. Here Ω is the angular velocity of the revolving cylinder and ν is the bulk kinematic coefficient of viscosity.

(iii) The velocity measurements are taken in the region $|x| \leq \sqrt{2ad}$ around the point of symmetry O, because the proposed equation is valid only within that region.

(iv) If formula (4.4) is chosen as an approximate basis for estimation, then the step of numerical differentiation h has to be much smaller than $\sqrt{2ad}$ (at a minimum, $h \leq 0.05\sqrt{2ad}$).

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