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NUMERICAL INVESTIGATION OF SEPARATED OR CAVITATING INVISCID FLOWS

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Studying the separation is one of the central issues of the contemporary hydrodynamics due to its outstanding importance in number of applications, e.g. cavitation, ship hydrodynamics, aerodynamics, etc. Development of theoretical approaches to separated and cavitating flows is obstructed by the formidable mathematical difficulties founded in the very nature of models of such flows, namely in the complex conjunction with the rigid boundaries of the separation line when in deneral the points of detachment/attachment are unknown.

The most frequently used method for solving free surface problems is perhaps that that of conformal mapping (see [1]) but it proved to be efficient chiefly for jet flows. For investigating the cavitating flows is used as a rule the method of integral equations (for survey see [2]). In the present work a direct difference solution is attempted making use of the approach developed in previous works [3],[4] where are treated free surface flows of ideal and viscous liquids, respectively.

1. Posing the problem. Consider the ideal flow around a long cylinder when the dependence on longitudinal coordinate can be neglected and the flow field can be considered two-dimensional. Let also for the sake of simplicity restrict ourselves to cylinders which possess line of symmetry. In terms of polar coordinates this line is definded as the junction of the two semi-infinite lines $\theta = \pi$ and $\theta = 0$. Respectively, the velocity at infinity is assumed to be parallel to the axis of symmetry so that to have the leading stagnation point at $\theta = \pi$ (see fig.1). The flow is pressumed irrotational and as so the stream function is to be govrned by the Laplace equation which in terms of polar coordinates reads:

(1) $\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} = 0$

with the following condition at infinity

(2) $\psi \sim r U_\infty \sin\theta$ for $r \to \infty$,

which is obtained as usual after trivial manipulations from the conditions for the velocity components.

The gist of the problem is in the boundary conditions at the combined boundary of body and cavitation zone (if the latter exists at all) which boundary is to be stream line. Without loosing the generality the number of this line can be set equal to zero

(3) $\psi = 0$ for $r = R(\theta)$, $0 \leqq \theta \leqq \pi$

where $R(\theta)$ is called henceforth shape function. Someportions of this function may coinside with the body shape function $R_b(\theta)$. Since we consider only relatively simple bodies with only one stagnation zone attached to the rear of the body (see fig.1) we pressume that stagnation zone occupies the interval $0 \leqq \theta \leqq \theta^*$.

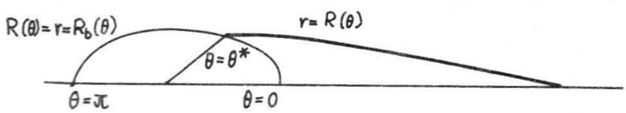

Fig.1

The problem is still not coupled since $R(\theta)$ is not known everywhere, and more specificly it is unknown in the stagnation zone. There the Bernoulli integral holds:

$$\left[\frac{2}{\rho}p + \frac{1}{r^2}(\psi_\theta)^2 + (\psi_r)^2\right]_{r=R(\theta)} = U_\infty^2 + \frac{2}{\rho}p_\infty \quad , \quad 0 \leqq \theta \leqq \theta^*$$

where p_∞ is the pressure at infinity and U_∞ is velocity at infinity. Introducing dimensionless variables according to following scheme

$$\psi = U_\infty L \psi' \quad , \quad r = L r' \quad , \quad p = p_c + \tfrac{1}{2}\rho U_\infty^2 q \quad .$$

where L is certain characteristic lenght and p_c is the pressure in the stagnation zone we get

(4) $\left[q + \frac{1}{r'^2}(\psi_\theta)^2 + (\psi_r)^2\right]_{r'=R'(\theta)} = 1 + \varkappa \quad . \quad 0 \leqq \theta \leqq \theta^*$

where

(5) $\varkappa = \dfrac{p_\infty - p_c}{\frac{1}{2}\rho U_\infty^2}$

is called cavitation number and for the flows of fluid with separation is equal to zero. For cavitating flows for which the pressure p_c is lesser than the pressure at infinity the cavitation number is positive.

Furthermore the primes shall be omitted without fear of confusion.

Eq.(4) consits of the dimensionless quantity q related to the pressure. On the boundary of separated zone this quantity is to be equal to zero as the pressure is equal to p_c due to the balance of normal stresses. So we arrive to following condition

(6) $$\left[\frac{1}{r^2}(\psi_\theta)^2+(\psi_r)^2\right]_{r=R(\theta)}=1+\varkappa \quad , \quad 0\leqq\theta\leqq\theta^*$$

from which the unknown function $R(\theta)$ has to be estimated implicitly.

2. Coordinate transformation. Boundary value problem (1), (2),(3),(6) offers two formidable difficulties when treated numerically. The first of them is that the boundaries of the region are not coordinate lines and the second one is that the unknown portion of the function $R(\theta)$ is to be identified implicitly in order to satisfy the Bernoulli integral. The most natural way to resolve the two difficulties is to introduce a new independent variable scaled by the shape function (see [3],[4])

(7) $$\eta=r/R(\theta)$$

Then the boundary becomes the coordinate line $\eta=1$ and Bernoulli integral (6) transforms into an explicit expression for shape function:

(8) $$\frac{1}{R^2(\theta)}\left[1+\frac{R'^2}{R^2}\right]T^2(\theta)=1+\varkappa \qquad \text{where } T\equiv\left.\frac{\partial\psi}{\partial\eta}\right|_{\eta=1}, \; 0\leqq\theta\leqq\theta^*.$$

As should have been expected the Laplace equation becomes somewhat more complex showing oblique derivatives but it proves to be a little worry for the numerical treatment

(9) $$\frac{1}{R^2}\frac{\partial^2\psi}{\partial\eta^2}+\frac{1}{\eta R^2}\frac{\partial\psi}{\partial\eta}+\frac{1}{\eta^2R^2}\left[\frac{\partial^2\psi}{\partial\theta^2}-2\eta\frac{R'}{R}\frac{\partial^2\psi}{\partial\eta\partial\theta}\right.$$

$$\left.+\eta^2\frac{R'^2}{R^2}\frac{\partial^2\psi}{\partial\eta^2}-\eta\frac{\partial}{\partial\theta}\left(\frac{R'}{R}\frac{\partial\psi}{\partial\eta}\right)+\frac{R'^2}{R^2}\eta\frac{\partial\psi}{\partial\eta}\right]=0.$$

The boundary conditions are recast into the new variables as follows

(10) $$\psi\sim\eta U_\infty R(\theta)\sin\theta \qquad \text{at } \eta\to\infty \; , \; 0\leqq\theta\leqq\pi ,$$

(11) $$\psi=0 \qquad \text{at } \eta=1 \; , \; 0\leqq\theta\leqq\pi .$$

It is important to derive in terms of new variables the expression for the exerted on the body force due to fluid motion. Without going in detail we obtain the expression:

$$(12) \qquad \vec{R} = \vec{\tau}\, \frac{\rho}{2} U_\infty^2 \cdot 2 \int_0^\pi \frac{1}{R} \sqrt{1+\frac{R'^2}{R^2}} \left[\cos\theta + \frac{R'}{R}\sin\theta\right] T(\theta)\, d\theta ,$$

where $\vec{\tau}$ is the unit vector in direction of axis of symmetry. There is nocomponent of the force which is normal to the axis of symmetry.

3. Difference scheme for Laplace equation. For the purposes of the numerical solution the infinite region $\eta \geqq 1$ is reduced to finite one $1 \leqq \eta \leqq \eta_{inf}$ where η_{inf} is certain sufficiently large number whose appropriate magnitude is estimated in numerical experiments. Uniform mesh is defined according to the following lows:

$$(13) \qquad \eta_i = 1 + (i-0.5)h_1 \quad \text{where } h_1 = (\eta_{inf} - 1)/(M-1) \qquad \text{for } i = 1, \ldots, M+1$$

$$(14) \qquad \theta_j = (j-1)h_2 \quad \text{where } h_2 = \pi/(N-1) \qquad \text{for } j = 1, \ldots, N,$$

where M and N are the total numbers of grid points in η and θ directions, respectively. The mesh is concerted in θ-direction but is shifted on half-step in η-direction (see fig.2) in order to get second order of approximation of the boundary conditions at infinity.

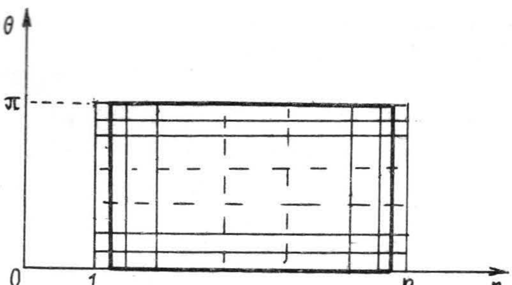

Fig.2

A fictitious time is introduced in Laplace equation and the splitting method of stabilizing correction (see [5]) is employed:

$$(15) \qquad \frac{\psi_{ij}^{n+1/2} - \psi_{ij}^{n}}{\tau/2} = (a_{ij}\Lambda_{11} + d_{ij}\Lambda_1)\psi_{ij}^{n+1/2} + (-2b_{ij}\Lambda_{12} + \Lambda_{22})\psi_{ij}^{n}$$

$$(16) \qquad \frac{\psi_{ij}^{n+1} - \psi_{ij}^{n+1/2}}{\tau/2} = \Lambda_{22}(\psi_{ij}^{n+1} - \psi_{ij}^{n}) \qquad \text{for } i = 2, \ldots, M,\ j = 2, \ldots, N-1$$

where

$$a_{ij} = \eta_i^2 \left[1 + \left(\frac{R_{j+1} - R_{j-1}}{2h_2 R_j}\right)\right], \quad b_{ij} = \eta_i \frac{R_{j+1} - R_{j-1}}{2h_2 R_j},$$

$$d_{ij} = \eta_i \left[1 - \frac{R_{j+1} - 2R_j + R_{j-1}}{h_2^2 R_j} + 2\left(\frac{R_{j+1} - R_{j-1}}{2h_2 R_j}\right)^2\right].$$

Scheme (15),(16) is coupled with the respective difference boundary conditions

$$(15^*) \qquad \psi_{1j}^{n+1/2} + \psi_{2j}^{n+1/2} = 0, \quad \psi_{M+1j}^{n+1/2} + \psi_{Mj}^{n+1/2} = \eta_{inf} R_j U_\infty \sin\theta_j \ , \ j = 2,\dots,N-1$$

$$(16^*) \qquad \psi_{iN}^{n+1} = \psi_{i1}^{n+1} = 0, \quad i = 2,\dots,M$$

Both the half-time step result into linear algebraic systems with tridiagonal matrices and are solved by means of a special kind of Gaussian elimination method with pivoting (see [6]).

The iterations are terminated when the following criterion is satisfied

$$(17) \qquad \max_{ij} \left| (\psi_{ij}^{n+1} - \psi_{ij}^{n}) / \psi_{ij}^{n+1} \right| \leqq 0.0001$$

and the obtained values of set functions are the consecutive global iteration, say ψ_{ij}^{α} .

4. Algorithm for calculating the shape function. Thinking of the stream function as known at α-th iteration stage one is to compute values of R_j from eq.(8). Unfortunately, that equation is not resolved for the derivative $R'(\theta)$. Moreover, it can be resolved only if certain condition is satisfied, namely if

$$(18) \qquad w(\theta) = (1+æ)\, R^2(\theta)/T^2(\theta) - 1 > 0.$$

Obviously, in the region adjacent to the leading or rear-end stagnation points eq.(18) holds because in those regions $T \to 0$ and therefore $(R/T)^2 \gg 1$. We discard from physical considerations the possibility to have separation near the leading stagnation point and focus our attention on the rear-end zone.

Computation of the α-th iteration begins with localization of the position of point at which $w(\theta)$ changes its sign from positive to negative, say it happens between the points with numbers j^* and j^*+1 . Then the following two steps are executed. Initially is solved (8) by means of following difference scheme

$$(19) \qquad R_{j+1}^{\alpha} - R_j^{\alpha} = h_2 R_j^{\alpha} \sqrt{(1+æ)\left(\frac{R_j^{\alpha-1} + R_{j+1}^{\alpha-1}}{T_j^{\alpha} + T_{j+1}^{\alpha}}\right)^2 - 1}$$

starting from initial condition $R_{j^*}^{\alpha} = 0$ and going backward to R_1^{α} . Secondly, the value $R_{j^*}^{\alpha}$ is improved through acknowledging the condition $R'(\theta^*) = R_b'(\theta^*)$ which is a direct corollary from (8). We satisfy this condition approximately at point θ_{j^*+1} taking three-point finite difference scheme for the derivatives and obtain

$$(20) \qquad R^{\alpha}_{j*} - R_{bj*} = 1/4\,(R^{\alpha}_{j*-1} - R_{bj*-1})$$

Thus the calculation of the α-th iteration of R is completed. The global iterations are terminated under the provisions of the following criterion

$$(21) \qquad \max_j |(R^{\alpha+1}_j - R^{\alpha}_j)/R^{\alpha+1}_j| \leqq 0.001$$

<u>5. Results and discussion.</u> The algorithm outlined in the above is employed to solve the problem of separated and cavitating flows around cylinders of circular or elliptic cross-sections. The essential features of this algorithm are that it does not require a specially devised closure at the tail of the cavity and that it is capable of automatic identification of the points of attachment of separation lines (surfaces). The former is due to a kind of numerical "closure" and means that the truncation error is so large and results in altering the properties of the free surface in the rear of the tail. The latter is the most important achievment of present work and means, in fact, that along with the potential one an other solution to Euler equations is found. The new solution consits of potential and stagnation parts which are mathched at unknown free surface.

On fig.3 are shown the shapes of cavities at circular cylinders for four different cavitation numbers

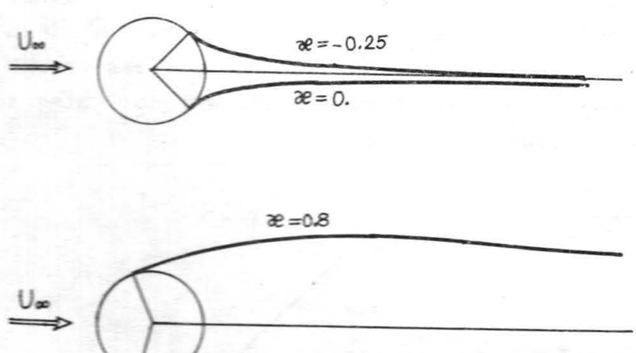

Fig.3

On fig.4 and fig.5 are presented results for elliptic cylinders with long axes parallel or normal to the flow, respectively. On fig. 6 is plotted the pressure distribution at circular cylinder for $\text{æ} = 0$. It is seen that in the stagnation zone ($\theta \leqq 40^{\circ}$) the pressure is equal to zero within the error of approximation, i.e. a flow of the descibed composite type (potential flow + stagnant zone) is indeed obtained.

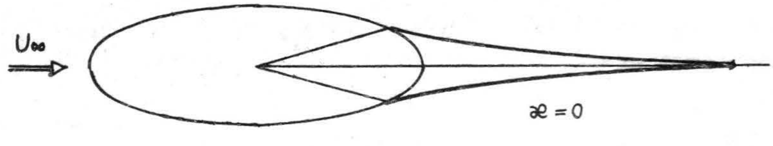

Fig.4

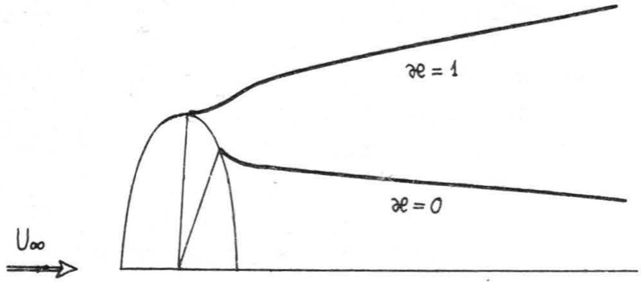

Fig.5

An important application of the method developed in the present paper is the prediction of the resistance of blunt bodies in large-Reynolds-number flows. On fig.7 is plotted the drag coefficient of circular cylinders versus Reynolds number. The resistance coefficient of separated ideal flow calculated here is represented by the dashed line. It is clearly seen that the calculated resistance compare quantiatively very well with experimental results for $Re > 10^5$. This means that the method developed can be used for at least rough estimation of the resistance of bodies of revolution without going to viscous or boundary-layer type calculations.

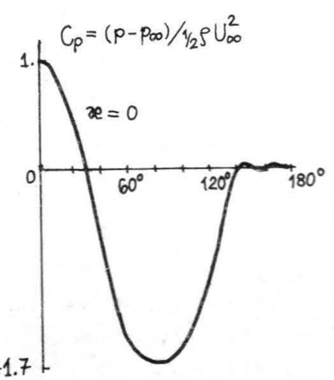

Fig.6

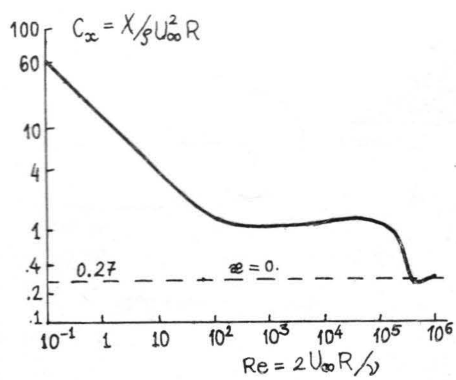

Fig.7

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