ON THE NUMERICAL TREATMENT OF SEPARATED UNSTEADY BOUNDARY LAYERS

I. Tzankov[1], C. Christov[2]

Bulgarian Academy of Sciences

Institute of Mechanics and Biomechanics

Department of Fluid Mechanics
Sofia 1090, P.O.Box 373, Bulgaria

ABSTRACT

The problem of numerical investigation of unsteady separated boundary layers is tackled by means of stable implicit splitting-type difference scheme. Non-uniform mesh is employed in normal direction and the algorithm is developed in general form. Its performance is checked by solving the Howarth's problem of boundary-layer flow in the rear-end stagnation point of yawed blunt body. The comparison, when possible, with the asymptotic solution is good, which speaks of the reliability of the proposed method.

INTRODUCTION

After Prandtl introduced it in 1908, the boundary-layer approximation turned out to be one of the most succesfull ideas of the modern fluid mechanics because of its simplicity and practical significance. The main advantage of stationary boundary-layer equations is that they are of parabolic type where the longitudinal coordinate plays the role of "temporal" coordinate. This allows one to construct marching methods for numerical evaluation of boundary-layer flows. Such methods are significantly less expensive in comparision with the methods for solving the elliptic systems with equivalent number of spatial coordinates. The very nature of this advantage, however, erects formidable obstacles on the way of treating the boundary layers with separation. In the last case the longitudinal velocity component becomes negative in the region of separation, which renders the governing equation to anti-parabolic type that is explosively unstable. This is the reason why the separated boundary layers are not commonly investigated numericaly. Only recently some attempts are undertaken to overcome the above mentioned difficulties (see [1] - [3]). In the present paper we outline a way to tackle this problem by employing the equations of the unsteady boundary layer. Hereafter we construct stable numerical method based on splitting procedure.

Unfortunately, the change of type is not the only deficiency of the boundary-layer equations when a separation is present. Even more severe a shortcoming is the presence of singularity in the point of reversion of the flow. This singularity was long ago pointed out by Goldstein and since that a unified point of view is not reached on the question whether the equations of boundary layer are at all applicable to reversed or separated flows (see, e.g. [1] - [3]). In numerical calculation the singularity shows up through a rapid growth of the boundary-layer thickness and eventualy through an instability of calculations conected with the thickness.

Guided by the notion that regardless to the answers of the above questions a reliable procedure for calculating the boundary-layer flows under unsteady loading oder in the presence of separation is needed anyway, we devote the present paper to this goal. As a featuring example we select the flow near rear - end stagnation point (see [6] -[12]) where a similar solution is available. In addition this is the most difficult task for every numerical procedure designated to treat boundary layers.

POSING THE PROBLEM

As the principal problem under consideration is not subject to the dimension of the space we confind ourselves to the case of two dimensions. The specific geometry of the flow is shown in Fig. 1. The parameters of the flow are the viscosity ν, the lenght of the plate L and the magnitude of the outer-flow velocity U_0 at the edge of the plate. Under the usual assumptions the boundary layer is governed by the Prandtl set of equations (see, e.g. [12])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\partial^2 u}{\partial y^2} \qquad (2)$$

*) This paper was presented at 13th Seminar on Ship Hydrodynamics, 1984

1 Postgraduate
2 Associate Professor, Dr

where u is the x-component of the velocity, v - the y-component, U - the outer-flow x-component. Respectively y coordinate is scaled by $\sqrt{Re}$, where

$$Re = \frac{L U_0}{\nu} \tag{3}$$

Respectively, x coordinate is scaled by L, and t - by L/U_0.

The type of the boundary-layer flow under consideration is specified by the choise of the outer-flow velocity $U(t,x)$. In the case of rear stagnation point one has

$$U(t,x) = 1 - x \tag{4}$$

The set (1),(2) is coupled by the respective boundary and initial conditions. The boundary conditions at the outer edge of the boundary layer and at plate are

$$u = v = 0 \text{ for } y = 0 \text{ and } u \to U(t,x) \text{ for } y \to \infty \tag{5}$$

The boundary (in fact - initial) conditions at the inlet of the flow $x = 0$ are subject to certain influence from the edges of the plate under consideration. It is well known, however, that the resulted disturbances to the boundary-layer flow are confinded in regions of lenght $O(Re^{-1/2})$ and can be discarded in the considerations (see, e.g. [11],[13],[14]). Thus for $x = 0$ we have

$$u = 0 \text{ for } y = 0 \quad \text{and} \quad u = 1 \text{ for } y \neq 0, \tag{6}$$

which is a discontinuous function of y, but this proves to be a little worry for the calculations.

We consider a body that is yawed from rest and this is modelled by a plate that is suddenly immersed in the flow. The latter drives us to the following initial conditions

$$u = 0 \text{ for } y = 0 \text{ and } u = U(x) \text{ for } y \neq 0. \tag{7}$$

Before proceeding furter it is interesting to mention some properties of the outlined boundary value problem. In particular, the asymptotic analysis gives that for short initial times of order $O(Re^{-1})$ the original problem is approximated by the following

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \tag{9}$$

with the boundary/initial conditions (4),(6). The important thing about the latter is that it does not require the boundary condition (5). Hence it might happen that a discontinuous solution for is obtained (see [13],[14]). The reason is that (8),(9) is valid only for the short times after yaw. Nevertheless, the solution of (8),(9) can be used for checking the accuracy of the difference scheme proposed below.

COORDINATE TRANSFORMATION

A crucial point for the numerical modeling is the implementation of the asymptotic boundary condition (4) for $y \to \infty$. In fact one always considers boundary layers whith finite thickness only and the said condition is fulfiled within certain prescribed accuracy. A new normal variable is introduced in place of y, according to $\eta = y/H(t,x)$, where $H(t,x)$ is called boundary-layer thickness. Thus the infinite region is rendered into rectangle which facilitates significantly the construction of the difference scheme and the algorithm.

In terms of the new normal coordinate the differential equation (2) adopt the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial \eta} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{1}{H^2}\frac{\partial^2 u}{\partial \eta^2}, \tag{10}$$

where for brevity is denoted

$$w = \frac{1}{H}\left(v - \frac{\partial H}{\partial t}\eta - \frac{\partial H}{\partial x}\eta u\right). \tag{11}$$

Respectively, the equation of continuity (1) transforms to

$$\frac{\partial w}{\partial \eta} = -\frac{1}{H}\left(\frac{\partial H}{\partial t} + \frac{\partial H u}{\partial x}\right). \tag{12}$$

The boundary and initial conditions are recast as follows

$$\begin{aligned} &u = w = 0 \text{ for } \eta = 0, \; u = U(x) \text{ for } \eta = 1; \\ &\text{for } x = 0 \quad u(\eta = 0) = 0, \; u(\eta \neq 0) = 1; \\ &\text{for } t = 0 \quad u(\eta = 0) = 0, \; u(\eta \neq 0) = U(x). \end{aligned} \tag{13}$$

The thckness $H(t,x)$ is to be evaluated from the condition for asymptotic matching of the boundary layer velocity $u(t,x,\eta)$ to the outer-flow velocity. The specific implementation of this notio is due below.

DIFFERENCE SCHEME

There exist a couple of numerical methods suitable for treating unsteady boundary layer when a limited reversed zone is present (see for survey [4],[5],[6]). The main difficulty is encountered when the nonlinear convective terms are approximated. Even one of the most used schemes called zig-zag [1]-[3] is not efficient enough because it is explicit to some extent and fails to satisfy the Courant condition of stability for arbitrary time increments. In the present paper is employed a fully implicit difference scheme of splitting type called in [15] a scheme of "stabilizing correction".

Let us denote for convenience the following notations for the operators in (10)

$$A = -u\frac{\partial}{\partial x}; \quad B = -w\frac{\partial}{\partial \eta} + \frac{1}{H^2}\frac{\partial^2}{\partial \eta^2}; \quad G(t,x) = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}. \tag{15}$$

The mesh is in general non-uniform

$$t_n = t_0 + \sum_{k=1}^{n} \tau_k; \quad x_i = x_0 + \sum_{k=1}^{i} \Delta x_i; \quad \eta_j = \sum_{k=1}^{j} \Delta \eta_k, \tag{16}$$

where τ_k are the time increments, Δx_i spacings in x-direction and $\Delta \eta_k$ - spacings in η-direction. The time stage for the set functions is denoted by the superscript "n", while the spatial position - by the pair subscripts "i,j". So the splitting scheme of "stabilizing correction" reads

$$\begin{aligned} &\frac{\bar{u}_{ij} - u_{ij}^{n-1}}{\tau_n} = A\bar{u}_{ij} + Bu_{ij}^{n-1} + G_{ij}; \\ &\frac{u_{ij}^{n} - \bar{u}_{ij}}{\tau_n} = B(u_{ij}^{n} - u_{ij}^{n-1}), \end{aligned} \tag{17}$$

where A and B are already certain difference approximations of the respective operators from (15). The half-time-stage quantity $\bar{u}$ is readily excluded and thus

$$(E - \tau_n A)(E - \tau_n B)\frac{u_{ij}^{n} - u_{ij}^{n-1}}{\tau_n} = (A + B)u_{ij}^{n} + G_{ij}, \tag{18}$$

which shows that the approximation is of order of $O(\tau_n)$.

Here is to be noted that in the limit $n \to \infty$, when $u^n \to u^{n-1}$, eq. (18) tends to the exact stationary equation and the time increment τ_n does not affect the approximation of the stationary problem. The last property is of crucial importance and most desirable when the solution of the steady case is intended.

Let us now turn to the difference approximation of operators A and B. Obviously, we need not go for a second- or higher-order approximation with respect to τ_n, since the accuracy of (18) is only $O(\tau_n)$. Hence we use the simplest quasi-linearization for the non-linear terms, namely

$$A = -u^{n-1}\frac{\partial}{\partial x}, \quad B = -w\frac{\partial}{\partial \eta} + \frac{1}{H^2}\frac{\partial^2}{\partial \eta^2}. \qquad (19)$$

The stability of difference scheme (17) and its order of approximation depend on the difference approximations of operators A and B. At the first half-time step is resolved the hyperbolic operator A and its difference approximation should comply with the requiremets of stability for hyperbolic schemes. Here we adopt the so-called upwind scheme (see, e.g. [16]).

$$u^{n-1}\left(\frac{\partial u}{\partial x}\right)^n \approx \begin{cases} u_{ij}^{n-1}\dfrac{u_{ij}^n - u_{i-1j}^n}{\Delta x_i} & \text{for } u_{ij}^{n-1} \geq 0 \\ u_{ij}^{n-1}\dfrac{u_{i+1j}^n - u_{ij}^n}{\Delta x_{i+1}} & \text{for } u_{ij}^{n-1} < 0 \end{cases} \qquad (20)$$

where the first index refers to coordinate x, and the second - to η.

The upwind scheme (20) is unconditionally stable, but its order of approximation is just $O(\Delta x)$. It should be stressed out, however, that increasing the order of approximation is possible only by compromising the stability. Another convenient feature of scheme (20) is that it takes into account the presense of zones of influence for the hyperbolic operator.

At the second half-time step is resolved the operator B whose difference approximation makes use of central differences

$$\frac{\partial^2 u^n}{\partial \eta^2} = \frac{2}{\Delta\eta_{j+1}(\Delta\eta_j + \Delta\eta_{j+1})}u_{ij+1}^n - \frac{2}{\Delta\eta_j \Delta\eta_{j+1}}u_{ij}^n + \frac{2}{\Delta\eta_j(\Delta\eta_j + \Delta\eta_{j+1})}u_{ij-1}^n + O(\Delta\eta_j \Delta\eta_{j+1}) \qquad (21)$$

and

$$\frac{\partial u^n}{\partial \eta} = \frac{\Delta\eta_j}{\Delta\eta_{j+1}(\Delta\eta_j + \Delta\eta_{j+1})}u_{ij+1}^n + \frac{\Delta\eta_{j+1} - \Delta\eta_j}{\Delta\eta_{j+1}\Delta\eta_j}u_{ij}^n + \frac{\Delta\eta_{j+1}}{\Delta\eta_j(\Delta\eta_j + \Delta\eta_{j+1})}u_{ij-1}^n + O(\Delta\eta_j \Delta\eta_{j+1}). \qquad (22)$$

Unfortunately, the second-order approximation (22) for the first derivative renders the scheme non-monotonous and the negative definitness of the difference analogue of B is true only when the normal velocity w is not very large. This is, however, the case in the problem under consideration and it did not prove necessary to take further messures for the approximation of the first derivative. Note that operator B yields to a three-diagonal linear algebraic system that is solved by means of Gaussian-elimination method with pivoting.

Before turning to the difference approximation of the equation of continuity it is necessary to discuss the problem with calculation of the boundary-layer thickness H. As it has been above mentioned the latter defined implicitly from the condition for asymptotic matching of the longitudinal velocity at the outer edge of boundary layer. The straightforward implementation of that condition, however, turned out to lead to computational instability connected with the unreliable calculation of the derivative $\partial u/\partial \eta$ at $\eta = 1$. We succeeded in constructing a stable algorithm only when we employed the following relation for defining H:

$$H_i^n = a H_i^{n-1}\int_0^1 (1 - u^{n-1}/U)\, d\eta, \quad a \doteq 6-8, \qquad (23)$$

i.e. the geometric thickness H_i^n at the n-th time stage is proportional to the displacement thickness calculated on the base of velocity profile of the (n-1)-th time stage. The coefficient a in (23) is at our disposal and we have selected its value in order to secure that the entire boundary layer lies within the geometric thickness H.

For the continuity equation we employ a scheme of second order of approximation with respect to both spatial coordinates and of first order with respect to time, namely

$$\frac{w_{ij+1}^n - w_{ij}^n}{\Delta\eta_{j+1}} = -\frac{1}{H_i^n}\left[\frac{H_i^n - H_i^{n-1}}{\tau_n} + H_{i+1}^n \frac{u_{i+1j+1}^n + u_{i+1j}^n}{2(\Delta x_i + \Delta x_{i+1})} - H_{i-1}^{n+1}\frac{u_{i-1j+1}^n + u_{i-1j}^n}{2(\Delta x_i + \Delta x_{i+1})}\right]. \qquad (24)$$

THE MESH PATTERN

The proper spacing of the non-uniform mesh is of crucial importance for the successful performance of the algorithm. Hence, the mesh should be suited to the specific problem as so to acknowledge the presense of regions of more steep dependence of the sought functions on the normal coordinate η. On this stage we are not yet able to adapt the mesh in the course of solution. Rather, we must make use of certain considerations and to construct mesh a priori. Fortunately, for the Howarth problem there exists abundant a priori information that allows us to conceptualize the main requirements for the grid.

The occurrence and development of reversed regions in the flow are characterized by the rapid growth of the boundary-layer thickness. In normal direction form two distinct regions (see e,g,[7],[8]), the first of them being adjacent to the rigid wall and remaining relatively conservative. In this region the modulus of velocity increases vigorously to the magnitude of the maximal reversion. The second region harbors the most important process of increasing of the velocity up to the magnitude of the outer velocity. In this region the velocity gradient varies from moderate to very small magnitudes. In other words, the picture is very similar to that of turbulent boundary layer and the first region roughly corresponds to the viscous sublayer.

Guided by the above considerations we selected the spacing in normal direction to be the following function

$$\eta_j = \frac{1}{\omega}\left(\frac{j-1}{N-1} + 0.4\right)\left[\exp\left(\frac{j-1}{N-1}\ln\frac{\omega+1.4}{1.4}\right) - 1\right] \qquad (25)$$

where N is the total number of grid points in normal dorection and ω governs the degree of

deviation from the uniformity.

RESULTS AND DISCUSSION

In order to check the performance of the devised numerical tool one has to conduct various numerical experiments involving different spatial spacings and time increments. For the present method the most important turned out to be the size of time increment and the mesh pattern in normal direction.

The first set of numerical experiments concern the mesh distribution in normal direction. Taking a relatively large time increment $\tau=0.04$ and a number of 51 points in longitudinal direction we run the calculations for the following three cases: 1) $N=101, \omega=10$; 2) $N=51, \omega=100$; 3) $N=51, \omega=10$. Only for the second set additional computations with $\tau=0.01$ and $\tau=0.02$ have been run. The results of these different computations appear to compare very well.

The computations are started from the natural boundary conditions (6) one of which is described by a discontinuous function. As it should have been expected the implicit difference scheme proposed possesses artificial viscosity and hence the initial disturbance (the discontinuity) is spread alongside with infinite velocity, while the physical considerations point out that the velocity of propagation should be finite (see [13], [14]). For this reason it is important to assess the role of the artificial scheme viscosity. In fig. 2 are presented the results obtained for the evolution of the solution during the first couple of time stages. It is seen that the downstream spreading of the conditions of the leading edge is negligible which means that the adopted mesh size is adequate. As should have been expected the derivatives of velocity, or which is the same - the drag forces, are more susceptible to the said artifical viscosity. Even for this quantity, however, the result for the roughest mesh ($N=51, \omega=10$) differs insignificantly from that obtained with the finest mesh. In ths instance, for the roughest mesh, a reverse current occurs at $t=0.66$, while with the more fine meshes the respective quantity is $t=0.64$ which coincides with the value obtained by other investigators [1], [9], [10].

Fig.3 depicts the downstream development of the wall viscous stress. The reversed flow takes place simultaneously in a whole region around the rear-end stagnation point. This result is in agreement with the observations in [7]-[9]. After the reversed region is formed the turning (separation) point moves with time upstream and for $t \approx 1.6$ reaches its stationary position at approximately 0.12 from the leading edge. As it has been above mentioned, this result was obtained originally by Howarth [11] (see also [12]). It is important to note that the same was obtained on the basis of the full set of Navier-Stokes equations [13]. The good comparison with the mentioned works means that the proposed here method can be used for reliable prediction of the separation point.

Table 1 gives insight of how the boundary layer at the every stagnation point evolves with time and the results obtained here are compared with [9], [10]. Once again one sees that the influence of the scheme viscosity is not significant, since the computations with different mesh sizes give essentialy the same result.

So far we have verified the ability of the proposed algorithm of adequate modelling of unsteady boundary layers. It is interesting now to investigate the problem of developing the singularity that has been spoken of above. It is expected that a jump of the displacement thickness occurs in the separation point [4]-[6], [10], [18]. For example, in [18] is shown that there exists a moment of time t_s such that when $t \to t_s$ hold $\delta \to \infty$ and $\partial\delta/\partial x \to \infty$. This result is supported from the similar finding for the Blasius-series method [10], [19]. At the same time the known numerical approaches either encounter irresistable difficulties long before t_s (see [2]), or do not predict any discernible changes of thickness at all (see [1]).

The dependence of the displacement thickness as a function of longitudinal coordinate is shown in fig.4 as obtained by means of the proposed numerical procedure. The characteristical local maximum observed in [10], [18], [19] appear in our calculation for $t \approx 1.7$. Its further numerical evolution speaks of an infinite increase of the thickness. Unfortunately, together with that increase the reliability of results diminish rapidly since raises the role of the fictitional viscosity which yields to smooting the peak at the separation point. Nevertheless, the results obtained here are not only in qualitative but also in quantitative agreement with the developing in recent years concept for behaviour of the separated boundary layers. In fact, the resuls presented are among the first quantitative numerical predictions for the unsteady boundary-layer flow in the presence of separation. The numerical technique developed can be applied also to more complicated cases of flow around blunt bodies and that is due in following papers.

REFERENCES

1. Cebeci T., The laminar boundary layer on a circular cylinder started impulsively from rest, J.Comput.Phys., 1979, v.31, pp.153-172
2. Telionis D.P., Tsahalis D.Th., Unsteady laminar separation over impulsively moved cylinders, Acta Astronautica, 1974, v.1, pp. 1487 -1505
3. Cebeci T., Carr L.W., Computation of unsteady turbulent boundary layers with flow reversal and evaluation of two separate turbulence models, Proc. Roy. Soc. London, 1982, A380, pp. 291-304
4. Telionis D.P., Unsteady viscous flows, New York/Heidelberg/Berlin, Springer - Verlag, 408p.
5. Smith F.T., On the high Reynolds number theory of laminar flows, IMA J.Appl.Math., 1982, v.28, pp. 207 281
6. Elliot J.W., Cowley S.J., Smith F.T., Breakdown of boundary layers (i) on moving surfaces, (ii) in semi-similar unsteady flow, (iii) in fully unsteady flow, Geophys. Astrophys. Fluid Dyn., 1983, v.25, pp. 77-138
7. Proudman I., Johnston K., Boundary layer growth near a rear stagnation point, JFM, 1962, v. 12, pp. 161-168
8. Robins A.J., Howarth J.A., Boundary-layer development at a two-dimensional rear stagnation point, JFM, 1972, v. 56, pp. 161-171
9. Hommel M.J., The laminar unsteady flow of a viscous fluid away from a plane stagnation point, JFM, 1983, v. 132, pp.407-416
10. Cowley S.J., Computer extension and analytic continuation of Blasius' expansion for impulsive flow past a circular cylinder, JFM, 1983, v.135, pp. 389-405

11. Howarth L., On the solution of the laminar boundary layer equations, Proc. Roy. Soc. London, A919, v.164, pp. 547-579
12. Шлихтинг Г., Теория пограничного слоя, 1969, Москва, "Наука", 742с.
13. Williams J.C., Rhyne T.B., Boundary layer development on a wedge impulsively set into motion, SIAM J.Appl.Math., v.38, 1980, pp. 215-224
14. Wang J.C.T., On the numerical methods for the singular parabolic equations in fluid dynamics, J.Comput.Phys., 1983, v.52, pp. 464-479
15. Яненко Н.Н., Метод дробных шагов, Новосибирск, "Наука", 1967
16. Réache P.J., Computational fluid dynamics, Hermosa Publishers, 1976
17. Briley W.R., A numerical study of laminar separation bubbles using Navier-Stokes equations, JFM, 1971, v.47, pp. 713-736
18. van Dommelen L.L., Shen S.F., The spontaneous generation of the singularity in a separating laminar boundary layer, J.Comput.Phys., 1980, v. 38, pp. 125-140
19. Ingham D.B., Unsteady separation, J.Comput.Phys., 1984, v.53, pp. 90-99

FIGURES

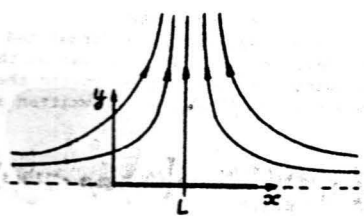

Fig.1

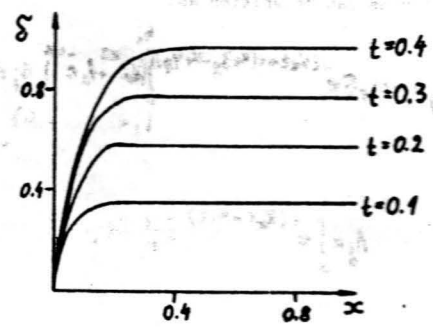

Fig.2. Evolution of the displacement thickness δ for small times.

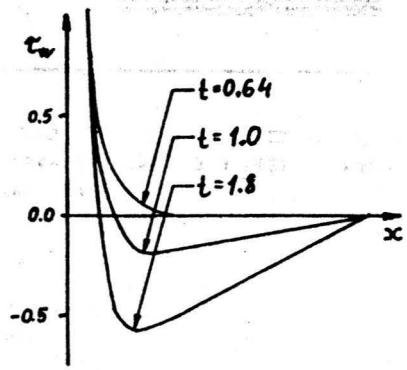

Fig.3. Evolution of the wall shear stress τ_w.

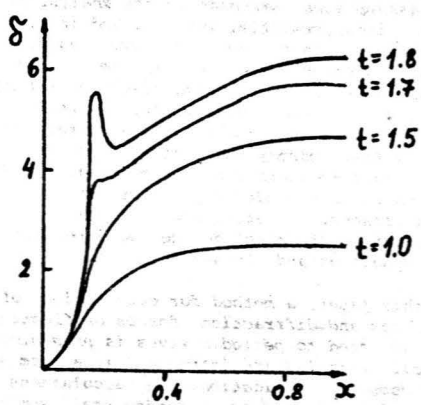

Fig.4. Evaluation of the displacement thickness δ for large times.

TABLE 1

	t	τ=0.02	τ=0.04	[6]	[7]
τ_w	0.5	-0.193	-0.204	-0.188	-0.188
	1.0	0.335	0.334	0.340	0.340
	1.5	0.658	0.658	0.666	0.666
	2.0	0.879	0.876	0.884	0.884
δ	0.5	1.19	1.18	1.196	1.196
	1.0	2.59	2.59	2.602	2.602
	1.5	4.72	4.73	4.719	4.720
	2.0	7.52	7.55	7.5	7.536