Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 42, № 1, 1989

GEOPHYSIQUE
Physique des l'atmosphère

ON THE BOUSSINESQ APPROXIMATION FOR LARGE SCALE ATMOSPHERIC MOTIONS

C. I. Christov, N. G. Godev

(Submitted by Corresponding Member S. Panchev on August 2, 1988)

Developing numerical models for atmosphere dynamics is one of the central problems of modern geophysical hydrodynamics. The basic difficulty on the way of doing that is the excessive generality of the equations which govern not only the large-scale motions of meteorological significance, but incorporate also the high-frequency acoustic and gravity waves. The latter imposes for stability of the difference schemes very stringent limitations stemming from the Courant-Friedrichs-Lewy condition [1]. For this reason a variety of systems of filtered equations has been developed (see [2] for a review). The problem is, however, that the most natural way of filtering, assuming an incompressibility can prove very rough and yield considerable inaccuracy for motions in which the dependence of density on temperature is crucial. Hence some authors employ the generalization of the incompressible model called Boussinesq [3] approximation in which the dependence of density on temperature is acknowledged in the momentum equations, while the continuity equation remains the same as in the case of incompressible liquid. The necessary improvements of Boussinesq approximation for atmosphere flows with intensive vertical motions can be found, e. g., in [4]. It is believed now that the Boussinesq approximation is suitable only for meso- and micro-scale motions. The present paper deals with rigorous posing the Boussinesq approximation for large scale motions, showing that some more terms are to be retained in the continuity equation.

1. The Original Set of Equations. Consider the general vertical form of governing equations referred to a coordinate frame firmly connected with the rotating Earth. The momentum (Euler) equations for an inviscid fluid read:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}, \nabla)\vec{u} + 2\,\Omega \times \vec{u} = -\frac{1}{\rho}\nabla P - g\nabla r$$

where $\vec{u}$ is the velocity in the moving frame, P — the pressure, Ω — the Earth's angular speed, ∇r — a vector, pointed outward from Earth's centre, g — gravity acceleration, ρ — the density of the air.

System (i) is coupled with the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{2}$$

and equation of state

$$P = \rho R T, \tag{3}$$

where R is the gas constant and T — the temperature. In what follows the temperature is assumed to be a known function as far as the equation of heat conduction. governs it.

Before doing anything we break as usual (see, e. g., [5,6]) the original system into two parts by means of the substitution

$$P = P_0(z) + P', \quad \rho = \rho_0(z) + \rho', \tag{4}$$

where P_0 and ρ_0 are the static values of pressure and density that are to satisfy the equations

$$\frac{\partial P_0}{\partial z} + g\,\rho_0(z) = 0, \quad \rho_0(z) = \frac{P_0(z)}{RT_0(z)}, \tag{5}$$

where z stands for the vertical coordinate measured outward along the Earth radius. It is worth mentioning here that $T_0(z)$, $P_0(z)$, $\rho_0(z)$ are also functions of the latitude which is not explicitly shown in (5) because the said functional dependence is of an order of magnitude weaker.

The characteristic scales of P_0 and ρ_0 are $[P] = 10^2\,\mathrm{kg.m^{-1}\,s^{-2}}$ and $[\rho] = 1\,.\,\mathrm{kg.m^{-3}}$, respectively. At the same time the dynamic part P' of the pressure is at most of an order of $\frac{1}{2}\rho U^2$, where $U \cong (20 \div 30)\,m\,.\,\mathrm{s}^{-1}$ is the characteristic scale of the geostrophic wind. Then $P' \cong 300\,\mathrm{kg.m^{-1}\,s^{-2}}$ and therefore

$$\left|\frac{P'}{P}\right| \cong 3 \times 10^{-3} \ll 1. \tag{6}$$

On the other hand, from (3) one obtains

$$\frac{\rho'}{\rho_0} = \frac{P'}{P_0} - \frac{T'}{T_0} + 0\left(\frac{P'}{P_0}, \frac{T'}{T_0}\right), \tag{7}$$

where T' is the disturbance of the static value of the temperature due to the dynamics. As far as $T_0 \cong 300^0\mathrm{K}$ and $T' < 30^0\mathrm{K}$ one has

$$\left|\frac{T'}{T_0}\right| \cong 10^{-2} \ll 1. \tag{8}$$

Neglecting the second- and higher-powers of (P'/P_0) and (ρ'/ρ_0) in the governing equations and being reminded of (5) one obtains:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\,.\,\nabla)\vec{u} + 2\Omega\times\vec{u} = -\frac{1}{\rho_0}\nabla P' + \left(\frac{1}{\rho_0^2}\nabla P_0\right)\rho', \tag{9}$$

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot[(\rho_0 + \rho')\vec{u}] = 0, \tag{10}$$

which is coupled by the linear approximation (7) to the equation of state (3). Once again the disturbance T' of temperature is thought of as a known quantity, because for it an equation can be derived.

2. The Boussinesq Approximation. As is above mentioned, the worst deficiency of the system (1)-(3) is concluded in the fact that the density ρ is a function of the pressure P, which renders it hyperbolic with a rather high speed of propagation of small disturbances (the speed of sound) and raises the respective limitations on the ratio between the time increment and spacings [1]. However, the said functional dependence can not be discarded in the original set of equations, because it is responsible for vertical distributions of density, pressure and temperature (the hydrostatics). The main idea is to do that only for the disturbances ρ' and P', instead for the original functions ρ and P. So, we simply neglect the first term in the right-hand side of (7) and arrive to the so-called Boussinesq approximation

$$(11) \qquad \rho' = -\frac{\rho_0}{T_0} T' \cong -\beta T', \text{ where } \alpha \cong 0.003 \cdot \frac{\text{kg}}{\text{m}^2 K^0}.$$

The discarded term is approximately of the same order of magnitude as the retained one, but almost three times smaller. Strictly speaking this is not enough as a justification to neglect it, but one can readily see that the distortion caused affects only the acoustic and gravity waves (simply eliminating them). In fact, relation (11) can be viewed as a version of (7) averaged over a sufficiently long time interval, that is much shorter than the characteristic time scale of evolution of T', but still extremely large in comparison with the acoustic time scale.

The Boussinesq approximation dramatically changes the type of the system and differs significantly from the other approaches that employ primitive equations (see, e. g. [5-8]). The gain is that we are faced now with a system which describes the motion of an incompressible though inhomogeneous fluid for which the continuity equation takes the form:

$$(12) \qquad \nabla \cdot [(\rho_0 + \rho')\vec{u}] = \Phi(x, t) \cong \alpha \frac{dT'}{dt},$$

where $\Phi(x, t)$ is a given function of the spatial coordinates x and time t. Respectively the Euler equations transform to:

$$(13) \qquad \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + 2\Omega \times \vec{u} = \frac{1}{\rho_0} \nabla P' - \left(\frac{\alpha}{\rho_0^2} \nabla P_0\right) T' + \left(-\frac{1}{\rho_0} \nabla P_0 - g \nabla r\right).$$

Due to (5), the last term in (13) is equal to zero in the equation for the normal component of velocity. In the equation for longitudinal component it once again contributes nothing and only the latitudinal equation is affected, because P_0 is a function of the latitudinal coordinate.

Eqs (12), (13) form a coupled system of equations for the unknowns $\vec{u}$, P, provided that the temperature disturbance T' is known. For solving the said system the well developed methods for treating incompressibe flows can be applied.

3. Scale Analysis. The form (12) of the continuity equation is the most general for the Boussinesq approximation. The terms in left-hand side account for all kinds of vertical scales for the convective processes (see [4]). However, the term in the right-hand side is being omitted in the known works without explicitly stating the reasoning for doing that. While due to the moderate scales of the flow in fluid-dynamics applications (see [3]) it is intuitively clear, in geophysical hydrodynamics the justifications for such an assumption are by no means obvious.

Let us denote by U ($\cong 20\ \text{m}^{-1}\ \text{s}$) the scale of horizontal components of velocity and by W — the scale of the vertical component. Respectively L (3×10^6 m) and H (2×10^4 m) are the horizontal and vertical scales. Consider a situation when the temperature increases (decreases) rapidly enough with time, say by 2^0K per hour. Then the characteristic scale for Φ is

$$[\Phi] \cong 0.003 \frac{2}{3600} \text{kg} \cdot \text{m}^{-3} \cdot \text{s}^{-1} \cong 2\times10^{-6} \cdot \text{kg} \cdot \text{m}^{-3} \cdot \text{s}^{-1},$$

and hence the eq. (12) yields the following connection

$$(14) \qquad \left\{\frac{LM}{DU}, 1\right\} = \left\{\frac{[\Phi]L}{[\rho]U}\right\} \cong 0.3.$$

It is obvious then that the right-hand side of (12) is of an order of unity and can not be neglected. The physical interpretation of the term Φ is obvious: it presents the additional divergence of the horizontal part of velocity field caused by the changes of density due to the unsteady heating (cooling). In regions of intensive heating (cooling) it creates source (sink) flows that under the action of Coriolis force turn to spiral motions. So, the term in (12) is a prime source of cyclogenesis.

An additional consequence of the presence of Φ in the right-hand side of (12) is that there are no more reasons to assume that WL/DU is of order of unity in order to satisfy the equation of continuity, i. e. one may not in general use $\operatorname{div}\vec{u}=0$ for defining the scale of vertical component (see [5, 7, 8] and others).

Conclusions. In the present paper the Boussinesq approximation allowing one to make use of the well developed methods for incompressible flows is rigorously posed for large-scale atmospheric motions. The dependence on pressure is neglected only for the deviation of the density from the statics. The scale analysis shows that in comparison to the classical Boussinesq approximation for large scales of motions another term is to be retained in the continuity equation which term is easily identified as 'thermal forcing' of the atmosphere due to the temporal changes of air temperature. Thus the proposed form of the governing system retains the major source of cyclogenesis (the classical Boussinesq approximation does not) of the full system of primitive equations being at the time decisively simpler. Namely, it is not yet hyperbolic which means that the high-frequency wave motions (gravity and acoustic waves) are completely switched off and the well known Courant-Friedrichs-Lewy condition limiting the time increment by the product of the inverse of sound speed and the spacings is eliminated.

Laboratory for Hydrometeorological Infomatics with Computer Centre Hydrometeorological Service, Bulgarian Academy of Sciences

Dept. of Atmospheric Physics Institute of Geophysics, Bulgarian Academy of Sciences

REFERENCES

[1] Courant, R., K. Friedrichs, H. Lewy. Math. Annalen, **100**, 1928, 32-74. [2] Haltiner G. J., R. T. Williams. In: Numerical Prediction and Dynamic Meteorology, Willey, New York, 1980 [3] Гершуни, Г. З., Е. М. Жуховицкий. В: Конвективная устойчивость несжимаемой жидкости Наука, Москва, 1972. [4] Panchev, S. In: Dynamic Meteorology, D. Riedel, Dordrecht, 1986. [5] Carpenter, K. M. Quart. J. Roy. Met. Soc. **105**, 1979, 629-655. [6] Browning, G. L., H. O. Kreiss Lect. Notes Physics, London, **264**, 1986, 128-133. [7] Washington, W., D. L. Williamson, Adv Res. Appl., **17**, General Circulation Models. Acad. Press, New York, 1977, 111-172 [8] Bates, J. R. Lect Appl. Math. 22, 1985, 1-29.