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# NUMERICAL INVESTIGATION OF THE BOUNDARY-LAYER FLOW AROUND IMPULSIVELY MOVED CYLINDER

**I. T. Tzankov, C. I. Christov***

*(Submitted by Corresponding Member S. Panchev on July 28, 1989)*

In recent years the problem of the existence of smooth solution to the unsteady boundary layer equations with unfavourable (adverse) pressure gradient has frequently been discussed in the literature where two distinct opinions are formed. The first one adheres to the notion that in finite time a singularity (not of Goldstein's type- develops for the shape of layer thickness leaving the wall shear stress regularly distributed. The second one is that such a singularity actually does not exist and the failure to obtain the approximate solution after certain times is attributed chiefly to the shortcomings of the numerical schemes employed. Reference is made to [1, 2] for a comprehensive review on the subject. Recently, by means of significantly different numerical methods (Lagrangian and Fourier-series) some authors, [3-5] have indicated the fact that a singularity develops with time for the solution of unsteady boundary layer equations. On the other hand the results in [6-9] obtained by means of Eulerian difference schemes are not in concert either among themselves or with the results of the previous group of works. They do not clearly answer the question of whether singularity exists or not.

In the present paper another scheme of this type is proposed and the respective numerical algorithm implementing it is developed. Ours is an implicit splitting-type scheme which is unconditionally stable. All mandatory measures for securing good approximation are taken, e. g. non-uniform mesh spacings in normal direction. A number of calculations with different magnitude of the longitudinal spacing were conducted in order to reveal the performance of the scheme in the vicinity of the separation point. The results obtained with the proposed robust Eulerian difference scheme are in good quantitative correlation with the results of the schemes of the first group [3, 5]. Hence the first conclusion of the present work is that the discrepancy observed until now between the Lagrangian and Eulerian schemes is not a matter of principle. The second conclusion supports the point of view that for unsteady boundary layer equations a singularity should be expected for finite time.

**1. Posing the problem.** Consider the two-dimensional viscous incompressible flow around a circular cylinder. In the usual notation, after appropriate choice of scaling we have to solve

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}+\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$u=v=0 \text{ for } y=0,\quad u\to U_e=\sin(x) \text{ for } y\to\infty, \tag{2}$$

$$t=0:\ u=0 \text{ for } y=0,\ u=U_e \text{ for } y>0. \tag{3}$$

The $y$-coordinate is rescaled with the function $H(t, x)$ of the outer edge of the computational region. The function $H(t, x)$ was set proportional to the displacement thickness $\delta(t, x)$ and the typical values of the coefficient of proportionality are from the interval (6, 8) which secures the asymptotic behaviour at $\eta = y/H \to 1$. Employing a scaled normal coordinate $\eta = y/H$, equation (1) is recast in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial \eta} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} + \frac{1}{H^2}\frac{\partial^2 u}{\partial \eta^2}, \quad \frac{\partial w}{\partial \eta} = -\frac{1}{H}\left[\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x}\right], \tag{4}$$

where

$$w = \frac{1}{H}\left(v - \frac{\partial H}{\partial t}\eta - \frac{\partial H}{\partial x}\eta u\right). \tag{5}$$

Boundary conditions (2), (3) are left by the transformation in their original form.

**2. Difference scheme.** To device an unconditionally stable difference scheme for numerical solution of boundary layer equations when the sign of longitudinal component of the velocity is positive is not a problem at all. The situation changes when a reversed zone is present, where the disturbances are convected opposite to the direction of the main flow. A comprehensive survey on the currently used approximations in the reversed zone has been performed in [1]. According to [8], all previous Eulerian difference schemes are only conditionally stable. This shortcoming of the employed numerical schemes is avoided in [10] by the use of iterations.

In the present work we employ the method of fractional steps, namely the scheme of Douglas and Rachford [11] see also [12]):

$$\frac{\bar{u} - u^{n-1}}{\tau} = A\bar{u} + Bu^{n-1} + f^n; \tag{6}$$

$$\frac{u^n - \bar{u}}{\tau} = B[u^n - u^{n-1}], \tag{7}$$

where $A$ and $B$ stand for the difference approximations $\mathbf{A}$ and $\mathbf{B}$, respectively:

$$\mathbf{A} = -u\frac{\partial}{\partial x}; \quad \mathbf{B} = -w\frac{\partial}{\partial \eta} + \frac{1}{H^2}\frac{\partial}{\partial \eta^2}; \quad f = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x}. \tag{8}$$

The approximation with respect to time can be assessed after excluding the half-time-step variable $\bar{u}$ (see [12]), namely

$$(E - \tau A)(E - \tau B)u^n = (E + \tau B)u^{n-1} - (E - \tau A)\tau B u^{n-1} + f^n \tag{9}$$

from which it follows that

$$(E + \tau^2 AB)\frac{u^n - u^{n-1}}{\tau} = (A + B)u^n + f^n. \tag{10}$$

It is easily seen now that the scheme is of first order of approximation with respect to time, implicit and unconditionally stable if the coefficients in operators $A$ and $B$ are 'frozen' and do not depend on the sought functions. The latter can be seen after eq. (9) is rewritten in the form

$$(E - \tau A)(E - \tau B)\frac{u^n - u^{n-1}}{\tau} = (A + B)u^{n-1} + f^n. \tag{11}$$

The differential operator $\mathbf{A}$ is antisymmetric, and $\mathbf{B}$ is negatively defined (see the definitions in (8)), i. e.

$$(\varphi, \mathbf{A}\varphi) = 0, \quad (\varphi, \mathbf{B}\varphi) \le -\gamma \|\varphi\|. \tag{12}$$

Then $\|E - \tau\mathbf{A}\| = 1$, $\|E - \tau\mathbf{B}\| \ge 1$, and therefore

$$\|[(E - \tau\mathbf{A})(E - \tau\mathbf{B})]^{-1}\| \le 1$$

which for the differential form of the splitting-type scheme is a sufficient condition for stability. The above property is retained also for the difference form provided that, after proper linearization, a conservative differencing for the set functions that secures the properties (12) is employed. All this means that the scheme is stable.

In the present paper the derivatives in **A** and **B** are approximated by central differences. Here is to be pointed out that for eq. (6) a fully implicit approximation can be devised that is independent of the sign of the longitudinal component of velocity. The resulting three-diagonal algebraic system is solved by Gaussian elimination with pivoting.

**3. Results and comparisons.** A number of numerical experiments have been conducted in order to assess the approximation of the scheme proposed and the performance of the algorithm. In a normal direction we took consecutively 26, 51 and 101 mesh points. The comparisons were made for the displacement thickness which is one of the most sensitive quantities in the problem under consideration, especially in the interval $[100° \leq x \leq 135°]$, where for larger times a non-monotonic behaviour develops. The results obtained with 51 and 101 points in normal direction differ by less than 1%. This accuracy is acceptable and therefore the chief portion of the calculations to be mentioned below were conducted with 51 points. The good accuracy attained in the present work on rougher meshes we attribute to the adequate choice of the non-uniform grid given by

$$\eta_j = \frac{1}{\omega}\left(\frac{j-1}{J-1} + 0.4\right)\left[\exp\left(\frac{j-1}{J-1}\ln\frac{\omega+1.4}{1.4}\right) - 1\right], \tag{13}$$

where $\omega$ is a parameter responsible for deviation of grid pattern from the uniform shape and $j$ is the number of the grid row.

Similar experiments were conducted with particular values of the time increment $\tau$ and mesh spacing in longitudinal direction $h$, and their influence on scheme approximation was assessed.

Turning to investigating the separation itself our results show that for $t \geq 2.5$ the displacement thickness $\delta$ grows fast and for approximately $t \approx 2.8$ a distinct local maximum is formed which is in good correlation with [3].

One of the most appropriate quantity for assessment of the separation is the displacement velocity $v_\infty = \partial(\delta U_e)/\partial x$. According to [5] for $t \to 3$ $v_\infty \approx (3-t)^{-1.75}$. On the other

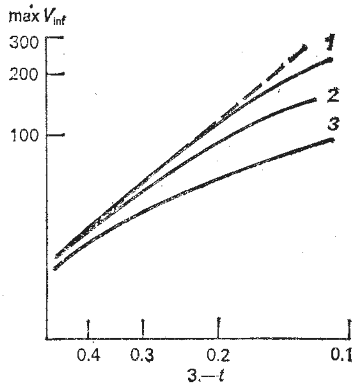


Fig. 1. Evolution of the displacement velocity when $t \to 3$: — — — — expected behaviour of the exact solution according to $\ln V_{inf} = -1.7414 \ln(3-t) + 1.9597$; 1 — $h = 0.625°$; 2 — $h = 1.25°$; 3 — $h = 2.5°$

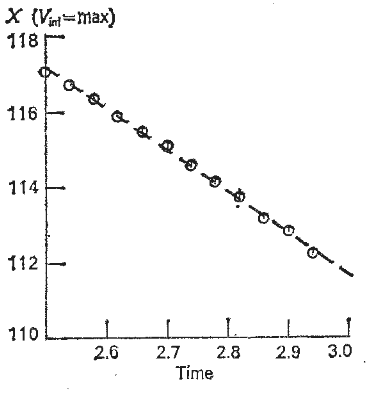


Fig. 2. Trajectory of the point of maximum displacement velocity: — — — — the linear approximation $x = 145.06 - 11.135\,t$; ○ ○ ○ — present numerical results

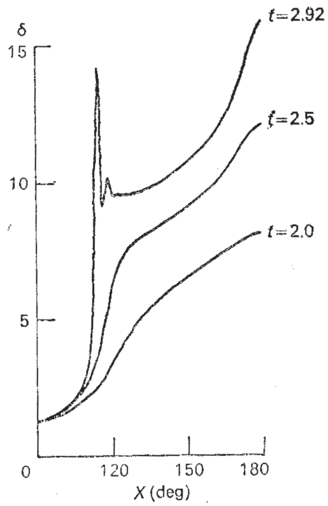


Fig. 3. Evolution of the shape of displacement thickness with time when $t \to 3$

hand, it is well known that the value of $v_\infty$ is a measure for the influence of the boundary layer on the outer potential flow. In the classical theory of boundary layers, when no separation is expected the displacement velocity behaves as $v_\infty \sim 0(\mathrm{Re}^{-1/2})$. Therefore the above suggested behaviour of $v_\infty$ speaks of singularity whose occurrence dismantles the classical theory. Fig. 1 shows the results for the maximal value of the displacement velocity obtained on three different meshes with number of points in longitudinal direction 73 ($h=2.5°$), 145 ($h=1.25°$), 289 ($h=0.625°$), respectively. The logarithmic scales adopted in Fig. 1 show the tendency of the computed curve to approach a line when $h \to 0$. Our results give 1.7414 for the slope of the line. The respective value from [13] is 1.75. Very close to this value are also the results in [4, 5].

Another important feature of the unsteady separation observed in [4, 13] is that the point of singularity of the solution moves alongside the wall in opposite direction to the main flow with constant velocity. Our results are plotted in Fig. 2 and they lie virtually on a straight line which for $t=3$ gives $\theta \approx 111.6°$. Let us note here that in [13] the reported value is $\theta \approx 111°$.

In the end it is instructive to obtain a sense of the separation process not only in the immediate vicinity of the separation point (the point of singularity) but also in the entire lee-side of the cylinder. For this reason Fig. 3 presents the displacement thickness in the said region.

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*Department of Fluid Mechanics*
*Institute of Mechanics and Biomechanics*
*Bulgarian Academy of Sciences*
*P. O. Box 373, 1090 Sofia, Bulgaria*

**Laboratory of Hydrometeorological Informatics*
*Hydrometeorological Service*
*Bulgarian Academy of Sciences*
*66 boul. Lenin, 1184 Sofia, Bulgaria*