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Splitting Scheme for Iterative Solution of Bi-Harmonic Equation. Application to 2D Navier–Stokes Problems

C. I. Christov* A. Ridha

Abstract

A ficititious time is introduced into the bi-harmonic equation rendering it to a higher-order (generalized) parabolic equation. The method of splitting of operator is used and shown that the splitting scheme approximates the generalized parabolic equation. In the half-time steps the splitting scheme reduces to 1D problem with five-diagonal band matrix. The test problem of steady viscous flow in lid-driven cavity is considered and results are obtained for the range $0 < Re < 40000$. The characteristics of the flow obtained here are in good agreement with the results of other authors.

The bi-harmonic equation appears in different fields of mechanics of continua, e.g., elasticity and viscous flows and developing efficient techniques for its numerical solution are of primary practical importance. It goes beyond the scope of the present paper to give extensive review of the literature on numerical approaches to Navier–Stokes equations. We refer the reader to the comprehensive review [3]. Here we concern ourselves with developing a fully implicit cost effective coordinate-splitting algorithm directly for the fourth-order equation for stream function. Thus the splitting method is applied as iterative procedure for solving the bi-harmonic boundary-value problems.

1 Posing the Problem

The 2D Navier–Stokes equations in terms of stream function reduce for steady flows to the nonlinear bi-harmonic problem

$$Re\left(\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right) = \Delta\Delta\psi \tag{1}$$

*On leave from the National Institute of Meteorology and Hydrology, Bulgarian Academy of Sciences, Sofia, BULGARIA

where $Re = UL/\nu$ is the Reynolds number based on the characteristic length scale L and velocity U. The dimensionless form of (1) is obtained after scaling the independent spatial variables by L, the stream function – by UL. The b.c. for an internal flow have the form

$$\psi = G(x,y)\,, \quad \frac{\partial \psi}{\partial n} = F(x,y)\,, \qquad \text{at} \qquad (x,y) \in \partial D \tag{2}$$

Here n stands for the outward normal to the region D and G, F are given functions, specified by the velocity components at the boundary. For stationary problems the boundary is a stream line and then without loosing the generality one may set $G \equiv 0$.

In the literature, the stream-function formulation was treated numerically by introducing the vorticity function $\omega = -\Delta\psi$ and rendering the problem to a coupled system of two second-order equations for ψ, ω. Although apparently more tractable, the $\psi - \omega$ formulation is inherently explicit, because of the lack of boundary condition for the vorticity function ω. Here we do not use the vorticity–stream function formulation, but rather solve directly the bi-harmonic equation for ψ. Thus we not only tackle the genaral problems of numerical solution of bi-harmonic equations, but also provide a method to overcome the inherent difficulties of the $\psi - \omega$ formulation.

2 Case Study: The Lid-driven Cavity Flow

As a rule, the new schemes and algorithms for solving Navier–Stokes equations are tested on certain test cases, most frequently on the case study of flow in a rectangular cavity driven by the steady horizontal motion of the lid. The advantage of this test case is that its geometry is the simplest possible. The disadvantage is that there are singularities at the points where the lid touches the vertical walls, but as it turns out the discontinuous boundary condition does not poses much difficulties to the algorithms.

The Chorin–Harlow method (MIC or fractional-step method) was applied to the lid-driven cavity flow in [6]. The latest progress along these lines can be found in [8] where the 3D problem of lid-driven cavity is treated. This kind of methods are by definition semi-explicit. The finite-element method was applied to the lid-driven cavity flow in [4] and results were obtained up to $Re = 5000$. In [11] the finite-element method was applied on adaptive grids and systematic results were obtained for $Re \leq 15000$.

The higher-accuracy methods in terms of vorticity–stream function were tested on the lid-driven cavity flow in [5] and [7], but due to the intrinsic instability of the said methods, reliable results were obtained in [5] only for $Re < 1000$ although the $Re = 2000$ is also mentioned, while in [7] the problems started showing up as early as for $Re = 300$.

The most-closely related to ours is the work of Schreiber & Keller [9] where the nonlinear bi-harmonic equation (1) was solved. They did not use neither the staggered mesh nor the coordinate splitting. They reported results for Re as high as 10000 but the higher Reynolds number were far from the values which are being now accepted as the solution to the problem. Their results compared reasonably with the other works only after Richardson extrapolation was applied. Yet, we believe, the work [9] is very important because it shows

that the the bi-harmonic problem is a good basis for numerical treatment of Navier–Stokes problems.

As a benchmark numerical paper for the flow under consideration is considered [1] where the $\psi - \omega$ formulation was used with explicit implementation of the condition on vorticity and upwind differences for the convective terms. Reliable results were obtained for $Re \leq 50000$ while the $Re = 10000$ was reached only after a smoothing operator was applied.

The square cavity occupies the situated in the domain $0 \leq x \leq 1, 0 \leq y \leq 1$ with the lid coinciding with the plane $y = 1$ and moving to the right in the said plane. Then the boundary conditions adopt the form

$$\psi = \frac{\partial \psi}{\partial x} = 0\,, \ \text{for } x = 0,1\,; \qquad \psi = \frac{\partial \psi}{\partial y} = 0, \ \text{for } y = 0\,, \quad \psi = 0, \ \frac{\partial \psi}{\partial y} = 1, \ \text{for } y = 1\,.$$

3 Difference Scheme

We add artificial time (say, s) in the equation for the stream function (1) rendering it into a generalized higher-order diffusion equation, namely

$$\frac{\partial \psi}{\partial s} = Re\left(\frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y}\right) - \Delta\Delta\psi \tag{3}$$

Since in the case under consideration the time is artificial, one need not worry about a second-order approximation with respect to it. The sole concern remains the stability of the scheme. In this instance we generalize the so-called "second Douglass scheme" or the scheme of "stabilizing correction" (see, e.g., [13] for terminology) to fourth-order operators as follows

$$\frac{\tilde\psi_{ij} - \psi_{ij}}{\sigma} = -\Lambda_{11}\tilde\psi_{ij} - \Lambda_{22}\psi^n_{ij} - \Lambda_{12}\psi^n_{ij} + F^n_{ij} \qquad \frac{\psi^{n+1}_{ij} - \tilde\psi_{ij}}{\sigma} = -\Lambda_{22}\left(\psi^{n+1}_{ij} - \psi^n_{ij}\right) \tag{4}$$

where σ is the increment with respect to the fictitious time s. Here $\Lambda_{11}, \Lambda_{22}$ and Λ_{12} are respectively the difference approximations of the fourth order derivatives with respect to x, y and the mixed fourth order derivative (see (9)), while F^n is a difference approximation of the nonlinear term (see (9)). Here the superscript n refers to the time stage: n is the current stage (the "old" one) and $n+1$ is the "new" one.

Denoting by E the unitary operator we recast the scheme as follows

$$(E + \sigma\Lambda_{11})\tilde\psi = (E - \sigma\Lambda_{22})\psi^n + (-\Lambda_{12}\psi^n + F^n)\sigma\,, \qquad (E + \sigma\Lambda_{22})\psi^{n+1} = \tilde\psi + \sigma\Lambda_{22}\psi^n\,. \tag{5}$$

After the fractional-step quantity $\tilde\psi$ is excluded one gets

$$(E + \sigma\Lambda_{11})(E + \sigma\Lambda_{22})\psi^{n+1} = (E + \sigma^2\Lambda_{11}\Lambda_{22})\psi^n + (-\Lambda_{12}\psi^n + F^n)\sigma \tag{6}$$

or which is the same

$$(E + \sigma^2\Lambda_{11}\Lambda_{22})\frac{\psi^{n+1} - \psi^n}{\sigma} = -(\Lambda_{11} + \Lambda_{22})\psi^{n+1} - \Lambda_{12}\psi^n + F^n \tag{7}$$

The last equality shows that the scheme in half-time steps (4) is equivalent to an implicit scheme in full-time steps. It is worth mentioning that the splitting scheme appears to be even more stable than an implicit scheme in full steps because of the presence of operator $\Lambda_{11}\Lambda_{22}$ which is positive definite (for commuting Λ_{11} and Λ_{22}) and hence the norm of the operator $(E+\sigma^2\Lambda_{11}\Lambda_{22})^{-1}$ is always lesser than unity.

An important feature of (7) is the explicit implementation of the nonlinear convective term and the mixed derivative term. As already mentioned, our main purpose is to propose an effective approximation of the viscous terms (the bi-harmonic operator). In such a case the most important is to have the diffusion terms represented implicitly, rather than the convective ones and this notion predestines our choice. In addition, in using the explicit scheme for convective terms we are spared the difficulties of their approximation on the "new" time stage notorious for generating artificial "scheme" viscosity, etc. In a sense, our treatment of convective term is a "purest" one that does not additionally pollute the overall $O(h_x^2+h_y^2)$ approximation of the scheme

In order not to obscure the main ideas we do not use here non-uniform meshes or other sophistications usually needed to improve the accuracy in the technological applications. The grid in both x- and y-directions is taken uniform, namely

$$x_i=-\frac{1}{2}h_x+(i-1)h_x\,,\quad i=1,\ldots,M+1\qquad y_j=-\frac{1}{2}h_j+(j-1)h_x\,,\quad j=1,\ldots,N+1$$

where h_x and h_y are the spacings in x- and y-directions, respectively. On the uniform grid we take the standard central-difference approximations:

$$\begin{aligned}
\Lambda_{11}\psi &\equiv \frac{1}{h_x^4}\left(\psi_{i+2,j}-4\psi_{i+1,j}+6\psi_{ij}-4\psi_{i-1,j}+\psi_{i-2,j}\right)\\
\Lambda_{22}\psi &\equiv \frac{1}{h_y^4}\left(\psi_{i,j+2}-4\psi_{i,j+1}+6\psi_{ij}-4\psi_{i,j-1}+\psi_{i,j-2}\right)\\
\Lambda_{12}\psi &\equiv \frac{2}{h_x^2h_y^2}\left[\psi_{i+1,j+1}-2\psi_{i,j+1}+\psi_{i-1,j+1}-2\left(\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}\right)\right.\\
&\qquad\left.+\psi_{i+1,j-1}-2\psi_{i,j-1}+\psi_{i-1,j-1}\right]
\end{aligned}\tag{8}$$

$$F_{ij}=\frac{Re}{4h_xh_y}\left[\left(\psi_{i,j+1}-\psi_{i,j-1}\right)\left(\omega_{i+1,j}-\omega_{i-1,j}\right)\left(\psi_{i+1,j}-\psi_{i-1,j}\right)\left(\omega_{i,j+1}-\omega_{i,j-1}\right)\right]\tag{9}$$

where

$$\omega_{ij}\equiv\frac{1}{h_x^2}\left(\psi_{i+1,j}+\psi_{i-1,j}-2\psi_{ij}\right)+\frac{1}{h_y^2}\left(\psi_{i,j+1}+\psi_{i,j-1}-2\psi_{ij}\right)\,,\tag{10}$$

is introduced only for the sake of convenience of notations.

At each half-time step the scheme reduces to a linear algebraic system with five-diagonal matrix. For instance, for the line $y=y_j$ one has to solve the following system for the vector $\psi_{1j},\ldots,\psi_{Mj}$:

$$\begin{pmatrix}
1 & 0 & 0 & & & & & & \\
0 & 1 & 0 & 0 & & & & & \\
1 & -4 & 6+\frac{h_x^4}{\sigma} & -4 & 1 & & & & \\
 & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \\
 & & 1 & -4 & 6+\frac{h_x^4}{\sigma} & -4 & 1 & & \\
 & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \\
 & & & & 1 & -4 & 6+\frac{h_x^4}{\sigma} & -4 & 1\\
 & & & & & 0 & 0 & 1 & 0\\
 & & & & & & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\\ \cdots\\ \psi_{ij}\\ \cdots\\ \psi_{M-1,j}\\ \psi_{Mj}\\ \psi_{M+1,j}\end{pmatrix}
=\begin{pmatrix}0\\ 0\\ b_{3j}\\ \cdots\\ b_{ij}\\ \cdots\\ b_{M-1,j}\\ 0\\ 0\end{pmatrix}\tag{11}$$

The iterations (time steps in fictitious time) are conducted until the satisfaction of the uniform norm criterion: $\max|\psi^{n+1}-\psi^n|<\epsilon$. After the calculations are completed the solution is "returned" to the non-staggered mesh according to the formula $\Psi_{ij}=\frac{1}{4}(\psi_{ij}+\psi_{i+1,j}+\psi_{i,j+1}+\psi_{i+1,j+1})$ and it is Ψ which is presented graphically

4 Results and Discussion

As has already been above mentioned, the main objective of the present work is to provide a fully implicit treatment of the bi-harmonic problem (the viscous terms) then the relevant case are the small and moderate Reynolds numbers.

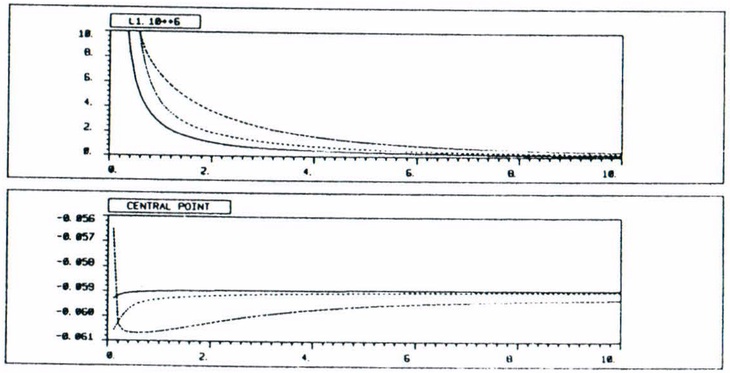

Figure 1: The evolution of solution and L_1-norm with the iterations for Re=1 for three different time increments: 1. $\sigma=0.002$ (solid line); 2. $\sigma=0.02$ (short dashes; 3. $\sigma=0.2$ (long dashes)

First of all, we show that the approximation of the steady solution does not depend of the fictitious-time increment of the splitting scheme. Theoretically it follows from eq. (7)

and provides a good check for the correctness of the algorithm if it is respected in practice. We calculated the flow with $Re = 1$ with three different fictitious-time increments: $\sigma = 0.002, 0.02, 0.2$. The evolution of the solution in the centre of the cavity (point $x = 0.5, y = 0.5$) is presented in Fig. 1 for a spatial resolution of 82×82 points and juxtaposed it to the evolution of the L_1 norm. We deliberately selected the case $\sigma = 0.2$, because in the virtue of $O(\sigma)$ approximation of the scheme one expects very rough approximation for large σ. Indeed, it is the case for the transient, but not for the steady state solution. In fact the cases $\sigma = 0.002$ and $\sigma = 0.2$ differ by less than 0.1% for the same number of iterations. It is clear then that the overall approximation of the steady problem does not depend on the time increment. The general feature here is the initial "overshooting" of the maximal value and consecutive monotonous convergence "from above".

The second important verification is the spatial approximation of the scheme. The case of $Re = 1$ is a trivial in this sense, because there is almost no influence from the convective term. We did obtain also a solution with 162×162 points for the same time increment $\sigma = 0.02$ and the difference was indeed of order of $h^2 \approx 4.10^{-5}$. Much more interesting is the case of higher Reynolds numbers, because we claim that our scheme has no artificial viscosity due to the explicit central-difference representation of the convective term. For that reason we treated the case $Re = 2000$ with $\sigma = 0.2$ with the two different spatial meshes and the difference between the two solutions was strictly of the second order.

These tests suffice to claim the good convergence of the scheme and its adequate spatial approximation. A general feature that distinguish it from the convergence methods for second-order diffusion equations is the somewhat "nonlinear" convergence in the sense that on the initial stages the algorithm descends rapidly to the sought solution, but when it gets in a closer vicinity of the latter the rate of convergence decreases and the final refinement takes considerable number of iterations. Obviously, the fourth-order diffusion is faster than the second-order one when it comes to dissipating the initial condition (when the gradients of the residual are large), but slower when the gradients are small (when the algorithm is near the sought solution). As a result the algorithm is slower in arriving to the final, steady-state stage. It is clear that this is the price that must be paid for the other advantages gained by the present approach to the problem, e.g., full implicitness and strong stability features.

There is consensus now (see, e.g., [11]) that the first sights of the third secondary vortex in the left-upper corner of the cavity appear for $Re = 2000$. That is another reason for solving that case with two different spatial meshes (resolutions). In Fig. 2 are shown the flow patterns for $Re = 1, 1000, 2000, 5000$ as calculated with mesh size 82×82. It is seen that the embryo of the third vortex does indeed appear for $Re = 2000$ and it is present there for the high-resolution calculations too.

The highest Re for which we were able to reach a truly steady solution was $Re = 5000$. In Fig. 3 we present the evolution of the solution in the centre of the cavity and the behaviour of the norm with the number of iteration for $Re = 5000$. After an initially violent evolution the norm gradually subsides to the order of 10^{-9} and the amplitudes of pulsations of solution decrease while their period increases. This means that the solution is approaching to the steady state (the iterations converge). Then the pattern of the flow presented in the last of

Figs. 2 pertains indeed to a steady solution. In Table 1 we present the comparison of our results (denoted by $C - R$ to the literature results. We select the most robust and in a sense, most representative characteristics – the value of streamfunction in the center of the primary vortex and its coordinates. Table 1 gives the comparison. It is interesting to note the general tendency of our results being off the closest reference results by 0.4%. This can be attributed first to the absence of of an artificial viscosity in our scheme and second – to the fact that we solve directly a problem with fourth derivatives that is much more flexible than a problem with second derivatives. In fact, here the lowest-order artificial terms comming from the difference approximation are the sixth-order derivatives, while in $\psi - \omega$ formulations, the lowest-order are the fourth derivatives, save for the second derivatives which arise from the convective terms when approximated with upwind differences. It is natural that a fourth derivative is "stiffer" than a sixth one, i.e., the profile of stream function ψ is more flexible in our case and its maximum/minimum is expected to be higher/lower than in the case of vorticity-stream function formulation.

The high-Reynolds-number flows could not be the primary concern for a scheme aimed to provide correct treatment of the diffusion terms. It is also clear than one can never have a scheme performing equally well for the whole interval of Re starting from zero and going up to the realm of turbulent régimes. Yet, because of its strong implicitness, the algorithm developed here turns out very efficient also for high Reynolds numbers. Surprisingly enough, for large Re the stability is achieved with very large increments of the fictitious time. In other words, in order to tackle the high-Re cases one need to jump during a given time step as farther ahead in time as possible. This way, the destabilizing effect of the convective term is diminished. It would have been a sheer paradox, if the progress made by the scheme during one time step were proportional to the size of the time increment. However, it is not. Even for time steps of order of hundreds, the relative progress of the solution toward the steady state is very slow and in some cases the convergence requires millions of time steps. The general scenario for very large Re is as follows. The rough solution to the problem is obtained within several hundreds of iterations (fictitious-time steps) and then the algorithm oscillates around the sought solution. The differences then are so slight that are hardly discernible. We were able to discover this kind of behaviour only after we went for L_1-norms of order of 10^{-7}. It is quite deceiving to stop the iterations when the norm reaches, say 10^{-6}, although it might look perfectly natural to do that. This sophistication is needed because of the nature of the investigated flow which exhibits a prolonged (in Re) laminar region and delayed transition to turbulence. In fact, the "weakly chaotic" solutions we encounter for $10000 < Re < 40000$ are very far from what is nowadays known as fully developed turbulence.

5 Conclusions

In the present paper we have studied numerically the steady 2D Navier–Stokes equations in terms of stream function. A fictitious time is added in the fourth-order equation for the stream function and coordinate splitting is applied directly to the bi-harmonic diffusion operator. It is shown that the splitting scheme does approximate an implicit scheme in full-

Table 1: The maximal value of $-\psi(x,y)$ and the positions x_{max} and y_{max} where it occurs

Re	1	100	400	1000	2000	3200	5000
[1]		0.10342 0.6172 0.7344	0.11391 0.5547 0.6055	0.11793 0.5513 0.5625		0.12038 0.5165 0.5469	0.11897 0.5117 0.5352
[9]	0.10006 0.50000 0.76667	0.10330 0.61667 0.74167	0.11297 0.55714 0.60714	0.11603 0.52857 0.56419			
[6][1]	0.099	0.103	0.112	0.116[2]		0.115	0.112
[12][3]		0.1034 0.6188 0.7375	0.1136 0.5563 0.6000	0.1173 0.5438 0.5625	0.1116 0.5250 0.5500		0.0920 0.5125 [4] 0.5313
[5][5]	0.100027 0.5 0.775	0.1035 0.625 0.75	0.1128 0.55 0.60	0.1115 0.525 0.575			
[10][6]		0.1034		0.1176[7]		0.1065	0.1181
[2]			0.1198	0.1136			
[11]					0.123 0.531 0.553		
[7]	0.0991 0.5 0.775	0.1054 0.633 0.748	0.1098 0.556 0.633	0.1092 0.556 0.595	0.1068 0.5 0.556		
C–R	0.100181 0.5017 0.7650	0.10397 0.6198 0.7369	0.11432 0.5533 0.6021	0.11871 0.5346 0.5645	0.11223 0.5228 0.5507		0.12661 0.5062 0.5232

[1]65 × 65, [2]97 × 97, [3]321 × 321, [4]161 × 161, [5]fourth order, [6]128 × 128, [7]256 × 265.

time steps. Thus no vorticity function is needed and the problem of boundary conditions is solved directly in an implicit manner at each half–time step imposing both the boundary conditions for the stream function.

The test problem of lid–driven cavity flow is treated by the new scheme as a featuring example. Results are obtained for Reynolds numbers in the interval $0 \le Re \le 40000$. For the higher Reynolds numbers $10000 < Re < 40000$ a kind of chaotic bifurcation is observed and the solution becomes pulsatile, exhibiting small but not decaying aperiodic oscillations around the mean flow.

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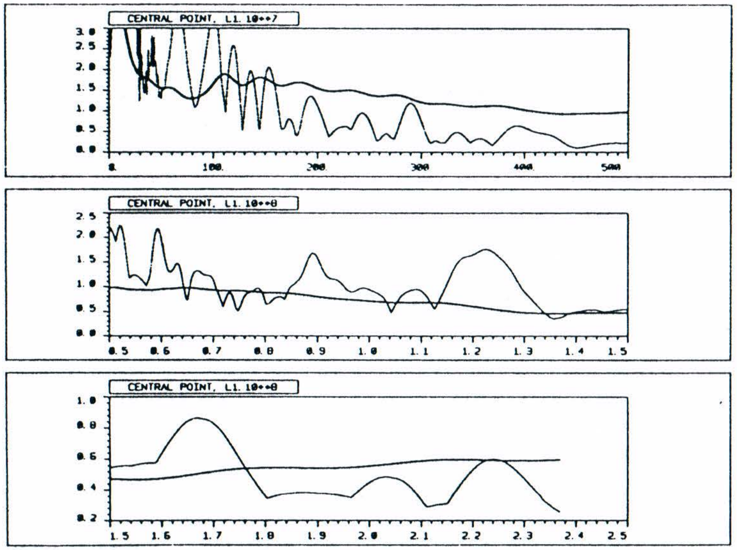

Figure 3: The evolution of solution and L_1-norm with the iterations for Re=5000. Solid line represents a scaled value of the stream function in the geometric center. The dashed line - scaled value of the L1-norm. In the upper figure the abscissa is measured in thousands of iterations; in lower two figures - in milions of iterations.