SOLITONS AND DISSIPATION

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ABSTRACT

In nonlinear dissipative systems waves can be created and maintained following an instability threshold. Here we describe a numerical exploration of their solitonic features appearing in collisions when appropriate balance exists between energy supply and dissipation.

1. Introduction

J. Scott Russell's discovery around "Turning Point" in Union Canal near Edinburgh and later on laboratory systematic experimental investigations on waves ([70]), see also[2,23]) led him "to pay great attention to a particular type which he called the 'solitary wave'. This is a wave consisting of a single elevation, of height not necessarily small compared with the depth of the fluid, which, if properly started, may travel for a considerable distance along a uniform canal, with little or no change of shape. Russell's 'solitary' type may be regarded as an extreme case of Stokes' oscillatory waves of permanent type, the wave–length being great compared with the depth of the canal, so that the widely separated elevations are practically independent of one another[75,76]. The methods of approximation employed by Stokes become, however, unsuitable when the wave–length much exceeds the depth[49]

Russell's observations and conclusions were at first sight in conflict with Airy's wave theory[1] where "a wave of finite height of length great compared with the depth must inevitably suffer a continual change of form as it advances, the changes being the more rapid the greater the elevation above the undisturbed level"[49] . It was not until the works of Boussinesq[5,7,6] and Lord Rayleigh[66] (see also[57]) that due credit was given to Russell's findings. Indeed, Boussinesq provided a theory (see also[20,43,66]) where higher–order linear dispersion was introduced in the long–wave limit and the crucial role of the balance between nonlinearity and dispersion was shown albeit not clearly seen by later authors (for an illuminating paper on the subject, see[83]). Particularly relevant is that Boussinesq found analytical expression of *sech* type for the permanent long–wave–length waves which are solutions of the equation he derived.

Later on, exploring consistently the simplifications of the problem under the assumption of slow evolution in the moving frame (already in Boussinesq derivations)

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and quoting the work of Boussinesq and Rayleigh, Korteweg and de Vries[45] derived an evolution equation governing the wave profile called now Korteweg–de Vries equation (or KdVE). For the slowly evolving in the moving frame (quasistationary) waves the Boussinesq equation strictly reduces to KdVE in the right–moving coordinate frame. Naturally, the same analytical solution of *sech* type is then valid also for the KdVE and that was, perhaps, the solution which has received the greatest attention during the last two decades. This bias is also the result of great discoveries made by Zabusky and Kruskal[91] on the one hand, and Gardiner et al.[30] and Lax[50], on the other.

Korteweg and de Vries found also another solution – the *cnoidal* wave train – that at two appropriate limits reduces to the Boussinesq solitary wave and to the harmonic wave train, respectively. (For accounts of the approximations leading to KdVE and related matters see[9,32,71,83].)

In the years 1952–53 Fermi, Pasta and Ulam[27] (referred further as FPU), with the help of Mary Tsingou and the MANIAC I computer of Los Alamos National Laboratory (USA), investigated the "energy–sharing" properties of the linear normal modes in certain nonlinear lattice models. They were interested in the process of relaxation to thermal equilibrium. Nonlinear lattices with cubic, quartic or broken–linear nonlinearity seemed to be ideal systems for such purpose. Relaxation of some non–equilibrium distribution back to the equipartition of energy in the interacting modes was expected because of the presumed ergodic nature of a nonlinear lattice (today we know that this assumption is not correct. For a detailed discussion of the FPU problem in modern context and an overview of nonlinear problems, solitons and all that see, e.g.,[3,10]). The question which FPU sought to investigate was how long it takes for a finite chain (e.g., with 16, 32 and 64 atoms), to reach the relaxation to equilibrium. The original objective had been to see at what rate the energy of the chain, initially put into a single mode like a harmonic wave, would gradually become a "mess" both in the form of the chain and in the way the energy is distributed among the modes. If all the energy is initially in the first normal mode one would expect the (nonlinear) coupling to generate a (slow) energy flow into the higher modes until equipartition with some small fluctuation is achieved. As $N \to \infty$ these fluctuations should vanish. Rather than that, what FPU discovered was recurrence to initial state. Fermi felt that their result "... constituted a minor/little discovery" although he intended to talk about this as a prestigious invited speaker (Gibbs lecture) at the annual meeting of the American Mathematical Society, as Ulam later recounted[27]. Fermi died (November, 1954) prior to the writing of their report (May, 1955). Although Ulam did talk about their findings at various scientific meetings, this report was only published, as a Los Alamos report (1955)[27] and in the open literature years later in Fermi's collected works (1965)[27]. It was the same year that several authors returned to the same problem and realized the major importance of such "minor/little discovery". It was realized that a continuum limit of the nonlinear lattice studied by FPU led to the long–time forgotten Korteweg–de Vries equation (KdVE)[45,84] which was known to possess solitary waves and (nonlinear) cnoidal wave trains very much like experimentally observed by J. Scott Russell in August 1834[70].

Following the track explored by Fermi and collaborators, in 1965 N.J.Zabusky and

M.D.Kruskal[91,90] reported on their numerical studies of the dynamical behaviour of the KdVE. Their motivation referred to a collisionless plasma which does not involve dissipation the latter being brought into the picture by the collisions. On the other hand, for lattices, the thermodynamic limit or "realistic" interaction with boundaries also brings irreversibility. They found that KdVE not only sustains solitary propagating disturbances, but that large-amplitude waves tend to break up into a spatial series of pulses with different amplitudes and velocities.

The second discovery of Zabusky and Kruskal was that these pulses retained their identity after they interacted, i.e., when they went through one another (or absorbed and emitted) they recovered their initial shapes. The only effect of their interaction is a shift in their space-time lines, corresponding to a temporary "acceleration". During their interaction their "combined" amplitude appears to be smaller than the sum of the initial amplitudes (linear superposition). Thus Zabusky and Kruskal coined the term *soliton* to describe a solitary, uniformly propagating localized disturbance (pulse), which preserves its structure and velocity after an interaction with another soliton, i.e. solitons had the property of ideal stable particles. When initially the energy resides in a "soliton", it will always remain in that state and will not be shared or thermalized[91,90]. Solitons in a nonlinear system are the analogues of normal modes of linear problems.

At about the same time (1965 onward) Visscher and collaborators[63,62,68] considered wave propagation in a nonlinear lattice, and using a Lennard-Jones interaction potential studied energy transfer in thermal conduction. For one- and two-dimensional lattices with many impurities of different masses they found that energy transfer was generally enhanced by the introduction of nonlinear interaction terms. Such result can be understood by assuming that energy is accumulated in the form of localized packets /pulses or solitons which propagate without being hindered much by the impurities. Thus lattices and continuum models (some nonlinear PDE's) were linked together in the opening of a new area in Physics and in Mathematics[30].

A special place of honor occupies Toda's exponential lattice, which in one limit yields the hard sphere gas and in another – the harmonic crystal [80,81,78,79,77]. It is the first many-body problem constructed and exactly solved using KdVE related elliptic functions.

Today we understand how under certain conditions nonlinearity and dispersion balance each other and localized solutions, stationary propagating waves (permanent solitary waves) take place in a dissipationless medium. The discovery goes back to Lagrange, Boussinesq[5,7,6] and Rayleigh[66] (see, however[83]). The KdVE was the first to undergo numerical investigation[91,90] unravelling the particle-like behaviour of the localized solutions. Since then a variety of conservation properties have been proved and a good deal of analytical techniques for solitons is now available for the KdVE and other soliton-bearing, integrable equations[24,78,79]. Thus the KdVE was not just an isolated curiosity.

Completely different is the case when dissipation is taken into account even as a small perturbation to the original model. The presence of dissipation spoils immediately integrability of the model and little can be done analytically, but who cares about integrability[72]? Yet this is the smaller evil. The problem is that very often dissipation is represented by higher-order spatial derivatives which means that considering it as a perturbation yields singular expansions that are only valid either for short times or on short distances. As far as the long transients (practically permanent in the time scale ε^{-1}) or stationary solutions are concerned the higher-order derivatives can not be treated as small perturbations. These are the obstacles on the way of systematic treatment and exhaustive investigation of dissipative systems, e.g., the possible solitonic behaviour of their localized solutions or of the crests in wave trains such as the periodic cnoidal wave. The importance of the self-sustained localized solutions (coherent structures – as called in the field of turbulence) stems from the fact that they can retain their individuality for considerably longer time intervals than the characteristic times of the small-scale disturbances. Studying their shapes and dynamics can provide insight into the properties of the particular system under consideration.

The main purpose of the present work is to show that the (input-dissipation) energy balance can add to the balance between nonlinearity and dispersion and to sustain particle-like solutions of dissipative Nonlinear Evolution Equations (NEE). This attitude is substantiated by a case study of the wave régimes of a nonlinear equation modelling the capillary flows in thin liquid layers – a dissipation modified Korteweg-de Vries equation. Mathematically speaking this equation is intrinsically dissipative – a generalized parabolic equation because of the highest-order derivative present. However, in ε^{-1} times it is much closer to a wave equation for pulse propagation in dissipative fluids and some reaction-diffusion systems in chemistry and related fields. On the other hand we envisage following the steps of Zabusky and Kruskal extending the physical meaning of the ideal *soliton* to the imperfect case (in the sense of van der Waals) in dissipative systems when one certainly does not care about integrability[72]. This is of interest in view of recently available experimental results[53,54,87,88]

2. A Dissipation-Modified KdV Equation

2.1. Heuristic Background and a Model Problem

The capillary flow in thin viscous films falling down an inclined plane (e.g., vertical wall) appears to be a model case for a continuous dynamical system which has low-dimensional phase space. Under the assumption of thin layer the flow in the bulk can be considered laminar and the influence of the bulk is reduced to the coupled drag force acting upon the surface. Thus an approximate equation containing only the surface variables can be derived for the elevation of the free surface. Since the works of Kapitza[39,40,41], it is known that the thin-film flow exhibits all major types of behaviour: laminar, periodic and turbulent. The main point here is that the latter is a "surface turbulence" – a chaotic behaviour of the long capillary waves on the surface while the bulk is perfectly laminar and the Poiseuille flow takes place in it.

The flow is governed by the Reynolds and Weber numbers†

$$Re = \frac{UL}{\nu} , \qquad We = \frac{\sigma}{\rho g h_0^2} , \tag{1}$$

where σ is the surface tension, ρ – the density, g – the gravity acceleration, h_0 – the thickness of the undisturbed film, ν – kinematic coefficient of viscosity and U is the characteristic velocity. The wave number is defined as the ratio $\alpha = \frac{h_0}{L}$, where L is the characteristic wave length.

Retaining terms up to order $O(\alpha)$ (see[4,38,51,52,58]) Gjevik[31] specifies the different terms of Benney equation to obtain the following $(1+1)D$ NEE (we use the notations of[33]):

$$h_t + (h^3)_x + \alpha \left\{ \frac{6}{5} \left(h^6 - \frac{Re_c}{Re} h^3 \right) h_x + \frac{\alpha^2 We}{3} h^3 h_{xxx} \right\}_x = 0 . \tag{2}$$

where $Re_c = \frac{5}{6}\cot\beta$ accounts for the inclination of the layer on angle β. This equation can produce singularities in finite time ([65,69])

If the characteristic length of the solution under consideration is not long enough, the term responsible for the surface tension contains also geometrical nonlinearities inherent in the expression for the curvature of the surface (see, e.g.,[38,61]). It was Homsy[33] who showed that in the weakly-nonlinear approximation (see, also the later works, e.g.,[46,74]), the consistent simplification of (2) gives

$$\eta_t + 3\eta\eta_x + \alpha Re \left\{ 3(\eta^2)_x + \frac{6}{5} \left(\frac{Rr - Re_c}{Re} \right) \eta_{xx} + \frac{P}{3} \eta_{xxxx} \right\} = 0 , \tag{3}$$

where $h = 1 + \alpha Re \eta$ and $P = \alpha^2 We$.

After re–scaling of variables the above equation can be recast as follows

$$\varphi_t + \varphi\varphi_x + \varphi_{xx} + \varphi_{xxxx} = 0 , \tag{4}$$

or

$$\psi_t + (\psi_x)^2 + \psi_{xx} + \psi_{xxxx} = 0 \quad \text{where} \quad \psi_x \equiv \varphi . \tag{5}$$

The last form of the NEE under consideration was also obtained in[48] for the evolution of reaction fronts for states close to a transition point and small wave amplitude. Nowadays the eqs.(3) or (4) are referred to as Kuramoto–Sivashinsky equation (KSE – for brevity). The only nonlinearity in KSE is the convective term (called in the framework of formulation (4) – "eikonal" nonlinearity[34,35]). Despite the assumptions valid in principle only near the threshold of instability, the model eqs.(3), (4) turns out to be suitable for nonlinear states well beyond the threshold (see[12]).

When the Marangoni effect is considered on the surface of the thin layer then an additional nonlinearity of the form $(\psi\psi_x)_x$ (see[29,44]) appears. The influence on the

†Nowadays we use the Bond number which is the inverse of the Weber number. The Weber number at present defines a balance between inertia and dissipation.

dynamical behaviour of this additional term as a destabilizing factor (according to its sign in the equation) was elucidated in[34]. On the other hand in a series of papers, Velarde and collaborators (see[28,59] and references therein) showed the consistent way of incorporating the Marangoni effect[86] into the one–way long–wave assumption and obtained for the case opposite to the Bénard convection[60,85], i.e., when heating the liquid layer from above (from the air side), the following equation[‡]

$$u_t + 2\alpha_1 u u_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (u u_x)_x = 0 \ , \tag{6}$$

containing both the dispersion term u_{xxx} (as KdVE does) and additional nonlinear term $(uu_x)_x$ accounting for the Marangoni effect. There is also an extension of this description to the 3D case with waves propagating in arbitrary directions[59]. We call in what follows the eq.(6) Korteweg–de Vries – Kuramoto–Sivashinsky–Velarde equation (KdV–KSVE). For the sake of completeness here we must mention that eq.(6) for the case $\alpha_5 = 0$ was considered in many other works (e.g., [25,26,42,47,55], etc., but with few exceptions only as a mathematical object without much concern about its physical relevance which shows the ubiquity of KdVE (Boussinesq Paradigm[20]) for description of shallow liquid layers and many other problems. The influence of the additional Marangoni nonlinearity on the the shapes of the solitary waves was investigated in[21].

Thus, although the derivation of eq.(6) provided by Velarde and colleagues[28,29] was part of a study of long–wave–length surface waves in shallow liquid layer subjected to Marangoni stresses due to imposed thermal gradients like with a shallow liquid layer heated from the ambient air or the derivation of other authors like Janiaud et al.[36] has a sound physical content in the study of Eckhaus instability for traveling waves we shall treat eq.(6) as mathematical object and hence as a model problem in itself. This out–of–specific–context approach to eq.(6) permits exploration of a wide class of solutions.

On the one hand eq.(6) which contains the original KdVE balance between nonlinearity and dispersion also captures the basic elements of a driven system with instability threshold and saturation elements, i.e., an energy balance with input–output (production–dissipation) which permits the onset and nonlinear evolution of a solitary wave decaying at both infinities or a shock (kink /bore /hydraulic jump) as well as solitonic wave trains.

On the other hand KSE as a mathematical object is prototype of an intrinsically dissipative system being in fact a generalized parabolic (heat–conduction) equation. The only difference here is that diffusion is represented in it by the fourth spatial derivative while the second spatial derivative has the opposite sign to the second–order diffusion equation, hence accounting for energy production (pumping). The same role plays also the Marangoni nonlinearity (when with the improper sign). This intricate interplay is characteristic for the long–wave instability of KS when the fourth order dissipation fails to bound the second–order pumping. Although apparently simple, theKSE possess a rich phenomenology (see[35]). As it will be seen henceforth,

‡when bottom friction is fully incorporated there also appears a term $\alpha_6 u$ which brings wave trains as the most relevant solutions[67]. It is not included here.

the considered here KdV–KSVE has even reacher one because in the short time scales it retains a lot of traits of a wave equation.

At this point it is worth mentioning that during the last decade, the nonstationary regimes of Rayleigh–Bénard buoyancy–driven convection in single and binary fluid layers heated from bellow or above were subject of numerous investigations (for review see Ref.[22]). The experiments performed first in rectangular geometries showed the existence of different kinds of motions, including counter–propagating waves, 'blinking' waves, etc. The situation was clarified when experiments were performed in narrow rectangular cells and especially in annular cylindrical geometries. The properties observed in nonstationary patterns agree qualitatively with theoretical predictions based on the complex Ginzburgh–Landau equation or its extension including the mass concentration field. Essential to the phenomena observed is the maintenance of a suitable level of external constraint with an appropriate balance with dissipation due to viscosity and heat or mass diffusion that sustains the stationary or nonstationary structure. Hence the obvious conclusion: with vanishing external constraint the reported phenomena disappear.

In the experimental conditions, to which eq.(2), (3), (4), or (6) correspond, because of strong stabilizing influence of the gravity and capillary forces on short wavelength surface deformations, the characteristic wavelength of the self–sustained surface waves is large in comparison with the layer depth. This is why such waves are describable by appropriate modifications of long wavelength equations known from the classical nonlinear theory. Indeed, the novel prediction of the recent studies, and a major difference with respect to the Rayleigh–Bénard buoyancy–driven nonstationary phenomena, is that these waves are not described by a complex Ginzburgh–Landau equation. We insist that eq.(6) is rather a generalization of a wave equation incorporating instability and dissipation. Hence, in the absence of dissipation and continuous energy supply, eq.(6) reduces to the standard KdV equation and solitary waves or cnoidal periodic wave trains are still possible. In experiment they can be excited from appropriate initial conditions either numerically[91] or in water tanks where viscosity plays negligible role[56].

Let us finally mention (see, Ref.[22]) some findings about liquid crystal/isotropic fluid interfaces of crystallization. Although there is reference to 'solitary waves', these waves correspond to finite trains of travelling waves resembling "confined states" observed in binary fluid convection. The interaction of such 'solitary' modes differs from the interaction of solitons of KdV or Boussinesq equations. As a rule, they coalesce, and in rare cases they annihilate. Further work is needed to assess whether those findings could be explained with a complex Ginzburgh–Landau equation or with a dissipation–modified wave equation.

2.2. Localized Solutions of KdV–KSVE and Energy Balance

We shall not consider here periodic wave trains but only solitary waves as 'localized' solutions. By definition the 'localized' solutions are supposed to asymptotically reach constant values at both infinities, namely:

$$u \xrightarrow{x\to-\infty} u_{-\infty} \qquad \text{and} \qquad u \xrightarrow{x\to+\infty} u_{+\infty} \,. \tag{7}$$

Due to their asymptotic nature these conditions yield trivial conditions for the spatial derivatives

$$u_x = u_{xx} = u_{xxx} \xrightarrow{x\to-\infty} 0 \qquad \text{and} \qquad u_x = u_{xx} = u_{xxx} \xrightarrow{x\to+\infty} 0 \,. \tag{8}$$

When stationary waves are considered in the moving frame $\xi = x - ct$, then (6) reduces to an ODE which can be integrated once to obtain

$$-cu + \alpha_1 u^2 + \alpha_2 u' + \alpha_3 u'' + \alpha_4 u''' + \alpha_5 u u' = C_1 \,, \tag{9}$$

where the prime stands for a differentiation with respect to ξ. The integration constant C_1 is defined from the asymptotic boundary conditions (6), (7) and must satisfy the following two conditions

$$-cu_{-\infty} + u_{-\infty}^2 = C_1 = -cu_{+\infty} + u_{+\infty}^2 \,, \tag{10}$$

They are consistent when

$$u_{-\infty} = u_{+\infty} = 0 \,, \tag{11}$$

which is the condition for 'humps': surface elevations/ positive solutions, or alternatively – dips/ depressions/ negative solutions, from an undisturbed level in the vicinity of origin of the coordinate system and asymptotically decay to zero at infinities. The hump is, in fact, the homoclinics of the ODE eq.(7).

When the boundary constants do not obey eq.(10) and do not adopt trivial values, the solution is called "kink" (bore/ hydraulic jump/ heteroclinics). In this case, the two conditions for the integration constant C_1 are consistent iff

$$c = \alpha_1(u_{-\infty} + u_{+\infty}) \qquad \text{and} \qquad C_1 = -\alpha_1 u_{-\infty} u_{+\infty} \,. \tag{12}$$

Before turning to the specifics of the numerical implementation it is important to address the issue of integral balances for the boundary value problems under consideration. It appears that those differ for the humps and kinks.

For the humps one can define the *mass*, and *energy* of wave as

$$M_h = \int_{-\infty}^{+\infty} u dx \quad E_h = \int_{-\infty}^{+\infty} \frac{1}{2} u^2 dx \,, \tag{13}$$

and thus

$$\frac{d}{dt} E_h = \int_{-\infty}^{+\infty} (\alpha_2 + \alpha_5 u) u_x^2 dx - \int_{-\infty}^{+\infty} \alpha_4 u_{xx}^2 dx \tag{14}$$

where we see that energy supply is significant at rather long wave lengths while the dissipation dominates at shorter ones.

The above definitions are inappropriate for the kink case, because then the integrals in definitions (12) diverge. In this case, we can define *mass* of wave as integral of the first spatial derivative (which has a hump shape), namely

$$M_k = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} dx \equiv u_{-\infty} - u_{\infty} \tag{15}$$

Respectively the *momentum* and *pseudoenergy* of the wave are defined as

$$P_k = \int_{-\infty}^{+\infty} \frac{1}{2} u_t = \alpha_1 (u_{-\infty}^2 - u_{\infty}^2) \equiv M_k c \qquad E_k \equiv \int_{-\infty}^{+\infty} \frac{1}{2} u_x^2 \, . \tag{16}$$

We use the terminology *pseudoenergy*, because the quantity E_k has no physical meaning as true energy of the wave, yet it is a quadratic functional associated with the latter. The "conservation laws" for the so defined *mass* and *momentum* of the wave are a matter of b.c. The balance law for the kink *pseudoenergy* reads

$$\frac{d}{dt} E_k = \int_{-\infty}^{+\infty} (\alpha_2 + \alpha_5 u) u_{xx}^2 dx - \int_{-\infty}^{+\infty} \alpha_4 u_{xxx}^2 dx + \int_{-\infty}^{+\infty} \alpha_1 u_x^3 dx \tag{17}$$

It is seen that for the *pseudoenergy* we have similar balance law, with the exception of the presence of the last term which could be either input or dissipation, depending on the particular shape of the wave.

3. Numerical Exploration of KdV–KSVE

3.1. Difference Scheme

It is convenient to render (6) into a system of two equations of second–order with respect to the spatial variable, namely

$$\begin{aligned} u_t + \alpha_1 u u_x + \alpha_5 (u u_x)_x + \alpha_2 q + \alpha_3 q_x + \alpha_4 q_{xx} &= 0 \\ u_{xx} &= q \, . \end{aligned} \tag{18}$$

The presentation as a system has significance for the numerical implementation allowing us to solve at each time step a system with better–conditioned matrix in comparison with the one that could have resulted from the direct difference approximation of the fourth–order equation. The details can be found in[16].

Consider the set functions $u_i, q_{i+\frac{1}{2}}$ on the regular mesh in the interval $[-L_1, L_2]$ with spacing h , i.e.,

$$x_i = -L_1 + (i-1)h \, , \qquad h = \frac{L_1 + L_2}{N-1} \, , \tag{19}$$

where N is the total number of grid points in the said interval. Note that the mesh for function q_i is staggered with respect to the main mesh where the function u is defined.

As far as we aim at the long–time evolution, it is preferable to have implicit scheme with very high order of approximation with respect to time in order to be able to use large time increments. For this reason we use a four–stage scheme which provides third order of approximation in time. As far as a multistage scheme requires more initial conditions than available, then at the initial stages it is simply two–stage or three-stage scheme. The latter means that we approximate the time derivative as follows

$$\frac{\delta u}{\delta t}\Big|_{i-\frac{1}{2}}^{n+1} \equiv \frac{1}{2}\Big[\frac{11u_i^{n+1} - 18u_i^n + 9u_i^{n-1} - 2u_i^{n-2}}{6\tau} + \frac{11u_{i-1}^{n+1} - 18u_{i-1}^n + 9_{i-1}^{n-1} - 2u_{i-1}^{n-2}}{6\tau}\Big] ,$$

$$\frac{\delta u}{\delta t}\Big|_{i-\frac{1}{2}}^{2} \equiv \frac{1}{2}\Big[\frac{3u_i^2 - 4u_i^1 + u_i^0}{2\tau} + \frac{3u_{i-1}^2 - 4u_{i-1}^1 + u_{i-1}^0}{2\tau}\Big] ,$$

$$\frac{\delta u}{\delta t}\Big|_{i-\frac{1}{2}}^{1} \equiv \frac{1}{2}\Big[\frac{u_i^1 - u_i^0}{2\tau} + \frac{u_{i-1}^1 - u_{i-1}^0}{2\tau}\Big] ,\tag{20}$$

Employing Newton's quasilinearization we arrive at the following difference scheme

$$\frac{\delta u}{\delta t}\Big|_{i-\frac{1}{2}}^{n+1} + \frac{\alpha_4}{h^2}(q_{i+\frac{1}{2}}^{n+1} - 2q_{i-\frac{3}{2}}^{n+1} + q_{i-\frac{3}{2}}^{n+1}) + \frac{\alpha_3}{2h}(q_{i+\frac{1}{2}}^{n+1} - q_{i-\frac{1}{2}}^{n+1}) + \alpha_2 q_{i-\frac{1}{2}}^{n+1}$$
$$+4\alpha_1 \frac{u_i^n u_i^{n+1} - u_{i-1}^n u_{i-1}^{n+1}}{h} = 2\alpha_1 \frac{u_i^{n^2} - u_{i-1}^{n^2}}{h} + O(\tau^3 + h^2)$$
$$i = 2, \ldots, N-2 , \tag{21}$$

$$\frac{1}{h^2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) = \frac{1}{2}(q_{i+\frac{1}{2}}^{n+1} + q_{i-\frac{1}{2}}^{n+1}) i = 2, \ldots, N-1 , \tag{22}$$

coupled with the following b.c.

$$u_1 = u_2 = u_{-\infty} , \quad u_{N-1} = u_N = u_\infty$$

3.2. Long–Time Transients of KdV–KSVE. "Aging" Solitons

As already pointed out, the immediate physical significance of the KdV–KSVE equation is for the case when the input and dissipation terms are smaller in comparison with the inertia and dispersion[28]. Through introducing appropriate scaling of the variables one can reduce the equation to one with $\alpha_2 = \alpha_4 = \varepsilon$ and $\alpha_3 = 1$. Respectively, the coefficient of the convective nonlinearity can be rendered to any value by means of re–scaling the amplitude of function u. For the sake of illustration we take $\alpha_1 = 3, \alpha_5 = 0$. The parameters to be varied are ε and the phase velocities of the structures imposed as initial condition.

One sees that the higher-order derivative of the equation under study is multiplied by a small parameter. The latter means that one is inevitably faced with a singular asymptotic expansion when one is to make profit of the smallness of the parameter. Then, the solutions of the dissipation modified equation should be expected to be reasonable well approximated by the solutions of the original KdVE only on a finite time interval of order of $O(\varepsilon^{-1})$. One of the objectives of the present Subsection is to verify this assertion.

Consider the well known *sech*-solution of KdVE:

$$u = \frac{3c}{2\alpha_1} sech^2 \left[\frac{1}{2} \sqrt{\frac{c}{\alpha_3}} (x - ct) \right] , \tag{23}$$

where c is the phase velocity of the *sech*.

The first important issue to be addressed when generalizing the notion of soliton to the long /long-time transients of the dissipation-modified KdVE, is the problem of collision of two localized solutions. We begin with the *sech* solutions of KdVE. The *sech* is not a solution of the full KdV-KSVE but for small ε the mismatch is supposed to be small and its impact should be felt at very large times, only (beyond ε^{-1}).

In Fig. 1 we present the over-taking collision of two *seches* of phase velocities $c = 10$ and $c = 5$, respectively. The result is very instructive in the sense that it confirms the intuitive expectation that on time intervals shorter than the dissipation time ε^{-1} the interaction must be essentially similar to the KdVE. Indeed Fig. 1-b shows the trajectories of centres of coherent structures. The dotted lines are projected trajectories of stationary propagating *seches*. The dashed line is the trajectory of the accelerating *sech* (see Fig. 2). It is seen that the phase shift is exactly the same as for the KdVE with the smaller *sech* experiencing the larger phase shift. The phase shift is "negative" in the sense that the structures are temporarily accelerated during the interaction and re-appear at positions farther ahead in the direction of their motion, than the position they would have reached, were the interaction not to take place.

Yet there is a conspicuous difference between the KdVE and the KdV-KSVE. In Fig. 1-a is seen that the total amplitude of the compound signal never exceeds the amplitude of the larger *sech*. It is well known form the the analytical two-soliton solution of KdVE that the amplitude of the compound signal is lesser than the sum of the amplitudes of the two initial solitons, but it is still considerably larger than the amplitude of the larger soliton. Thus the inelasticity of interaction of *seches* is significantly exaggerated by the presence of the KS part.

The most important finding is that for small $\varepsilon = \alpha_2 = \alpha_4$ a threshold of the initial amplitude can be defined according to which the coherent structure evolves into one or another permanent shape. Fig. 3 shows that the *sech* of phase velocity $c = 5$ (amplitude 2.5) was not "strong" enough to maintain the production-dissipation energy mechanism and after the interaction gradually decreases to a permanent shape of lesser amplitude. What is shown in Fig. 3 is the slow "aging" of the solitary wave attaining eventually the stationary shape within a dimensionless time of order of 10000 which is in fact 10 times larger than the inverse of the small parameter in this case.

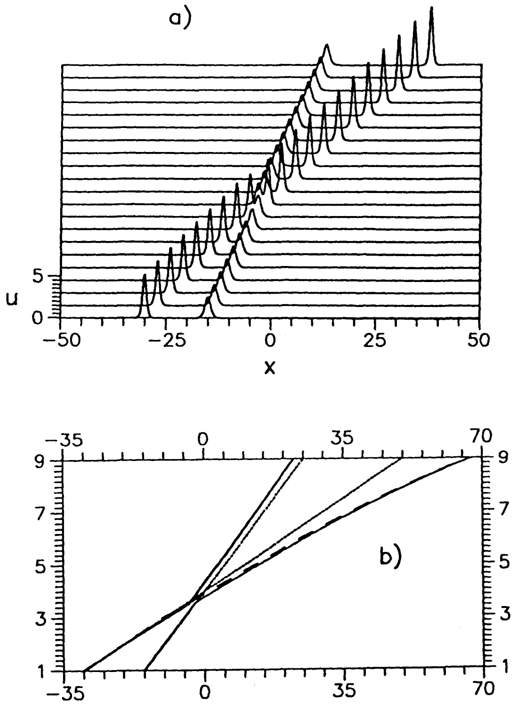


Figure 1: Interaction of two *seches* with $c = 10$ and $c = 5$ in KDV–KSVE for small KSV part $\alpha_2 = \alpha_4 = 0.001$: a) evolution of shape with time; b) trajectories of centres of solitons.

At the same time, for the more intensive *sech* from Fig. 2 of phase velocity $c = 10$ (amplitude 5), the shape change due to the interaction triggers the production and the *sech* starts growing until reaches the shape of larger amplitude that is a solution of the full KdV–KSVE equation (Fig. 2–a). It is interesting to note here that the process here is faster than the process of decrease of the previous case, being completed within a time of order 30, which is of order of $0.03\varepsilon^{-1}$. Since the KS part is multiplied by a small parameter, the equilibrium shape is not significantly different from a *sech* profile of the respective amplitude. Fig. 2–b shows both the profile of the coherent structure and its reflection with respect to the origin of coordinate system. It is clearly seen that deviation from the symmetric shape is very small. Yet the influence of the dissipation here is felt in the fact that there is a *selection* of the phase–velocity. The value which corresponds to the result presented in *Figs.*2 is $c = 17.64$.

Elphick et al.[25] mention the effect of selection and even give a numerical value for the eigen parameter – phase velocity for one set of parameters. The problem was

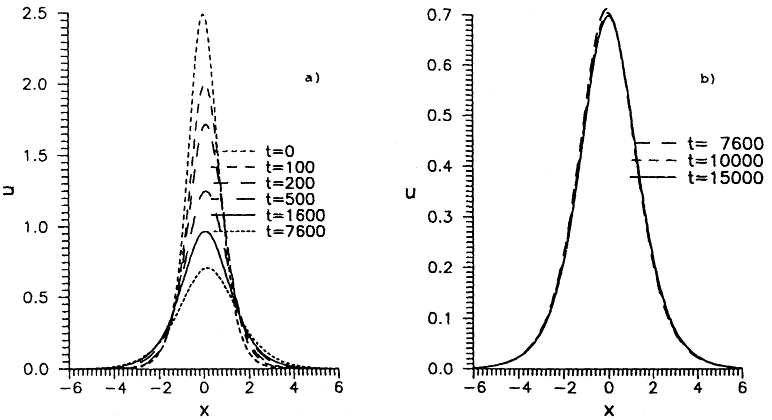


Figure 2: Growth of a *sech* with $c = 10$ in kdv–ksve for small KSV part $\alpha_2 = \alpha_4 = 0.001$: a) evolution of shape with time; b) asymmetry of the permanent shape.

given a special treatment in our previous investigation[15,21] where the eigen parameter was calculated with high accuracy for different sets of governing parameters. When a time dependent solution is considered being numerically shifted together with the moving frame, the accuracy could never be as high as for the specialized methods of solution[15,21] tailored to tackle the inverse nature of the problem. Yet, the solutions of present work show a very high degree of stationarity and the first three to four digits of the phase velocity are fully reliable.

In order to reveal the influence of the KS part we return again to the case of a single *sech* taken as an initial condition. In ***Figs***.4 are presented the evolutions for different values of $\varepsilon = \alpha_2 = \alpha_4$. Fig. 4–a depicts the case with small $\varepsilon = 0.02$, but not so small as the above considered case. The *sech* gradually decreases in amplitude and decelerates to the eigen value for the particular case $c = 6.79$. Some signals are created that escape in direction opposite to the direction of motion of the main *sech*. A unique feature for cases $\varepsilon \approx O(0.01)$ is the occurrence of a envelope which moves to the left with velocity of order of 40. This kind of signal is not observed neither for smaller, nor for larger values of ε (see, Fig. 3,2 and the rest of ***Figs***.4). This pattern is sustained by an appropriate balance between the production of energy, dispersion and nonlinearity. We have specially investigated the envelope and it turned out to be explosively unstable blowing up in finite time.

In Fig. 4–b the case $\varepsilon = 0.1$ is presented. The scale of the graph is changed in order to be better seen the signals that are created. One sees that once again, the main hump decelerates down to the eigen phase velocity of KdV–KSVE equation. The next to the main hump gradually accelerates to the eigen phase velocity and attains the permanent form (this is for times which are not presented in Fig. 4–b). The rest is a spatially decaying radiation propagating to the left. Qualitatively the

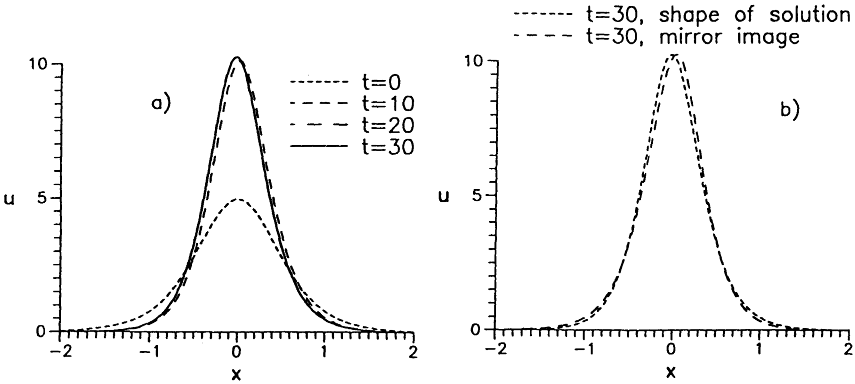
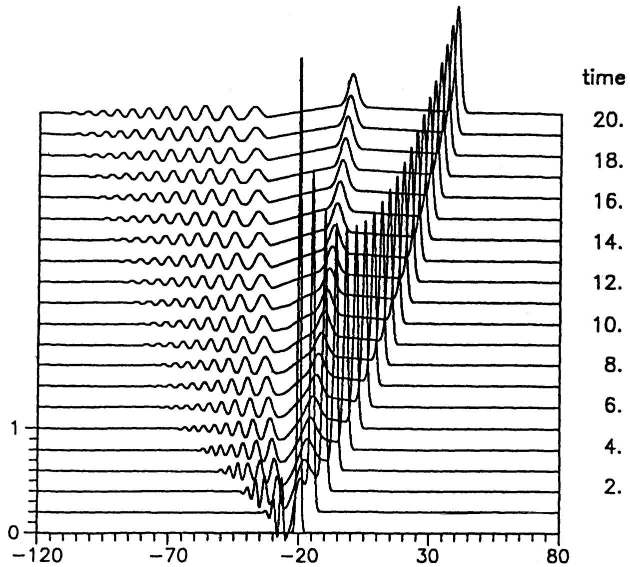


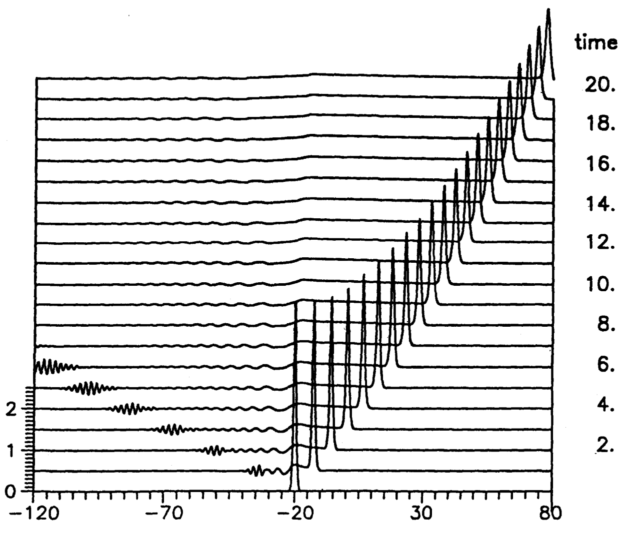
Figure 3: "Aging" of a *sech* with $c = 5$ in KDV–KSVE for small KS part $\alpha_2 = \alpha_4 = 0.001$.

same is the case $\varepsilon = 0.5$ shown in Fig. 4–c. However, another important feature is seen here, namely that the signal escaping to the left grows until its largest hump becomes large enough and begins an evolution on its own eventually reversing the direction of propagation and turning to the right. At the time the second hump is much faster now in attaining the permanent shape. This scenario is illustrated in Fig. 4–d for $\varepsilon = 2$. It is seen there how the consecutive humps of the left–going signal grow enough, reach a threshold and begin evolution as coherent structures that eventually attain the permanent shape. These pictures are convincing illustration of our conjecture that apart from the balance between nonlinearity and dispersion there exists another mechanism for sustaining the localized structure, namely the (production–dissipation) energy balance. The permanent shapes are those that satisfy such a balance. Beyond $\varepsilon = 2$ is situated the realm of chaotic solutions of the dissipative system under consideration. For the plain KSE this happens for arbitrary small values of ε because there is no dispersion there. One can say that dispersion plays a *laminarizing* role (see, also,[13,25,42]). In Fig. 4–e is presented the chaotic régime into which transforms a single *sech* for large $\varepsilon = 10$ when the behaviour is essentially dissipative (essentially KS–type). Although chaotic, the solution of the equation under consideration is dominated by the presence of large deterministic structures, in the sense that a significant part of the energy of pulsations is connected with these structures. This kind of intermittent chaotic/deterministic behaviour is acknowledged in turbulence, where the coinage "coherent structure" was introduced and methods of phase averaging were developed for pattern recognition of structures (see, e.g.,[8]).

An appropriate mathematical model for treating intermittent deterministic–chaotic régimes turned out to be the so–called random point functions that are random trains of similar shapes the latter being randomly distributed throughout the space or in time. The random point model allowed effective approximate closure of the cascade equations of turbulence and was instrumental in predicting the statistical characteris-

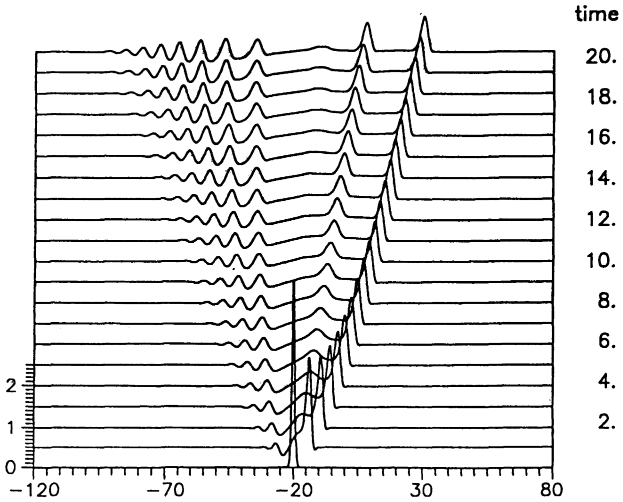


a) $\varepsilon = 0.02$;

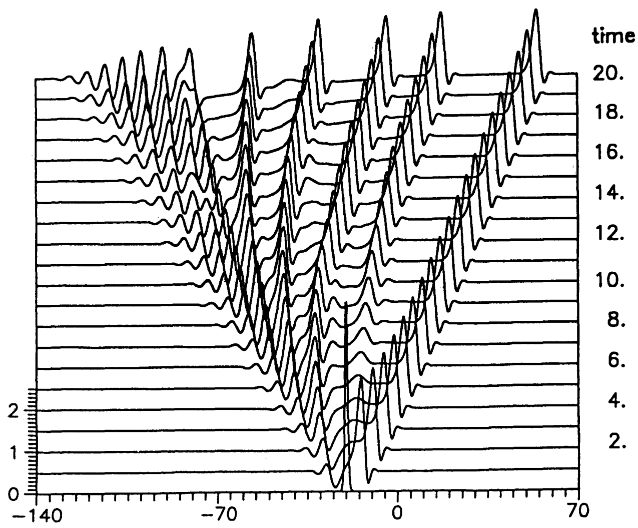


b) $\varepsilon = 0.1$;

Figure 4: Evolution of a *sech* with $c = 10$ in KDV–KSVE for different $\varepsilon = \alpha_2 = \alpha_4$:

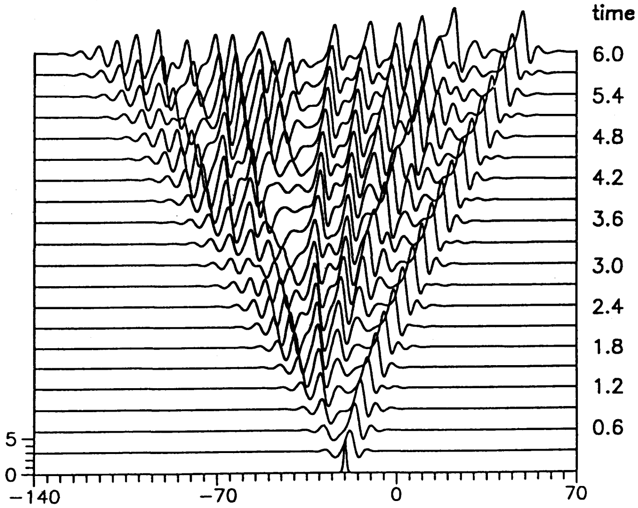


c) $\varepsilon = 0.5$;



d) $\varepsilon = 2.$;

Figure 4: Continuation



e) $\varepsilon = 10.0$.

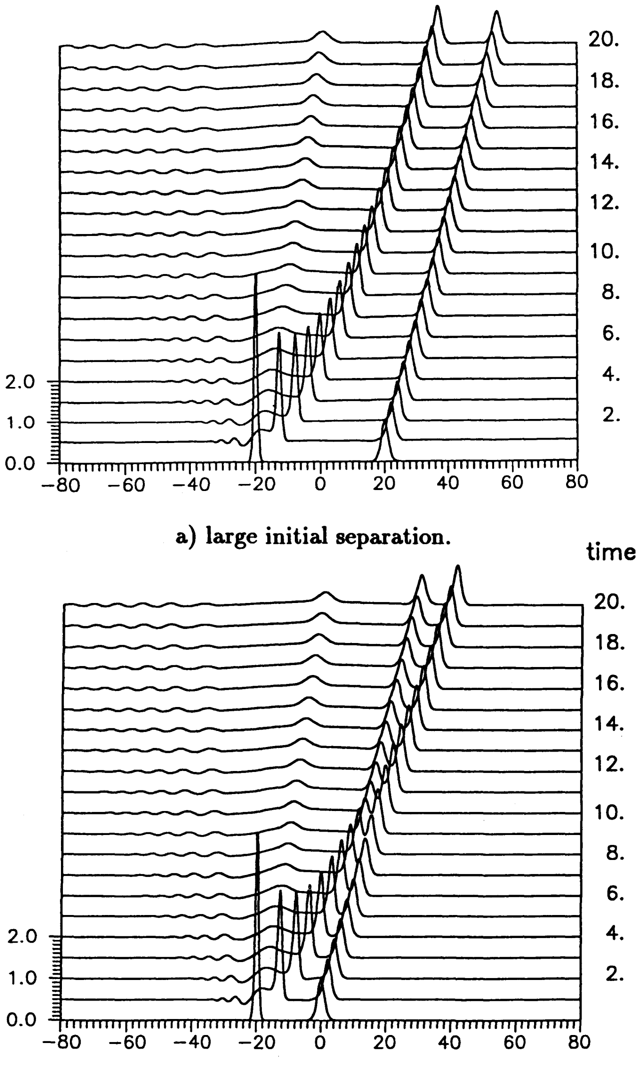
Figure 4: Continuation

tic for different chaotic motions, e.g., Lorenz attractor, Poiseuille flows, plane mixing layer[14,19]

As shown in[17], the chaotic solution of KSE pertains to the same class being fairly well approximated by a random point function. The predicted correlation functions, higher–order moments, etc. statistical characteristics were in very good quantitative agreement with the direct numerical experiment[73] (for more details, see also the monograph[19]).

Let us now resume investigating the overtaking collisions. In Figs. 5,6 we present the takeover of a small *sech* of initial phase velocity $c = 2$ by a large one with $c = 10$ for $\varepsilon = 0.1$. Taking such large value for the parameter ε allows us to observe significant effect within a shorter dimensionless time (and hence – reasonable computational time). In Fig. 5–a the two *seches* are separated enough and within the characteristic time $\varepsilon^{-1} = 0.1^{-1} = 10$ they could not catch with each other. Because of their "aging", they attain the permanent shape (and hence the terminal phase velocity) before they become close enough in order to interact. In fact the amplitude of the large soliton diminishes twice for dimensionless time equal to one! Approximately twice is the reduction of the phase velocity. It is obviously not an effect of order $\varepsilon = 0.1$. which is another facet of the singular asymptotic nature of the problem under consideration. Much less violent is the evolution of the smaller *sech* which is much closer the the eigen–solution of the stationary KdV–KSVE for this particular value of ε.

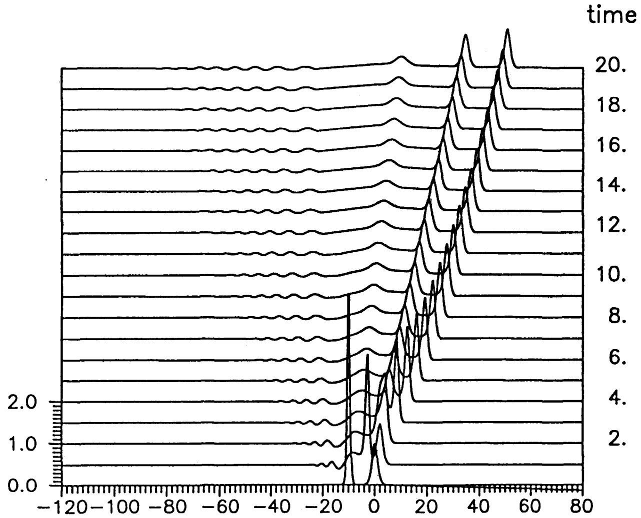
The next experiment is presented in Fig. 5–b where the initial separation of the



a) large initial separation.

b) moderate initial separation.

Figure 5: Overtaking interactions of a *sech* with $c = 10$ and $c = 2$ in KDV–KSCE for $\varepsilon = \alpha_2 = \alpha_4 = 0.1$



a) the collision stage;

Figure 6: The case of Fig. 5 for small initial separation;

seches is reduced twice in comparison with Fig. 5–a. This distance is short enough and even after the faster crest has lost half of its speed it is still able to get close enough to the slower one so that both feel each other. However, they can not pass through each other. In fact, they reach a distance when the "repelling" part of the interaction potential becomes significant and the *underdog* slows its motion even further because of the repulsive force form the forerunner. Then the energy of the former is transferred to the latter which grows and becomes somewhat faster. Then the two structure slowly separate to a distance compatible with one of the local minima of the potential of interaction (as said in[42] one structure "nests" in the local minimum of the other). By then, they are already of permanent shape and move precisely with the same phase velocity. Thus they form a *bound state* which is *metastable* in the sense that it can be broken only by strong enough disturbance but not by an infinitesimal one. Further reduction of the initial separation reveals a genuine clash of the two structure during which they form a single hump (see the line corresponding to t=2 in Fig. 6–a). Hereafter the scenario is the same as the case in Fig. 5–b. In order to illustrate what has been said about the bound state we proceeded with keeping track on the evolution of the signal from Fig. 6–a. In Fig. 6–b is seen the perfectly stable evolution of the solution whose speed is indeed of the order of the inverse of the small parameter. In Fig. 6–d is shown the solution for two different large times ($t = 4\varepsilon^{-1}; t = 6\varepsilon^{-1}$). The two large humps have already attained their terminal shape and phase velocity. The third hump is still growing at the time presented in the figure. It eventually reaches the same terminal speed and shape, but after a longer time.

For larger ε the terminal shape becomes more complicated exhibiting deeper local minima. Then one is to expect tighter[§] bound states. Indeed, the results for $\varepsilon = 1$ confirmed this supposition. The two structures form after the takeover a tight bound state that moves as a single coherent structure with steady phase velocity.

The same effect is observed also for $\varepsilon = 2.$, but the picture is much more complicated. As already mentioned, the value $\varepsilon = 2$ can be considered as the threshold of chaotic régime and beyond it the dynamics is richer. After the takeover different signals are excited and part of them form bound states with different degrees of "tightness" (Fig. 7–a). Especially instructive is the portion of the signal presented in Fig. 7–b where one can see three fully developed permanent shapes: a single hump and two two–hump bound states with different distances between the humps. Obviously, the tighter bound state is formed when the forerunner structure "nests" in a local minimum that is closer to the centre of the trailing structure. The exhaustive taxonomy of the bound states and the investigation of the margins of their stability (metastability) go well beyond the frame of the present work and will be presented elsewhere.

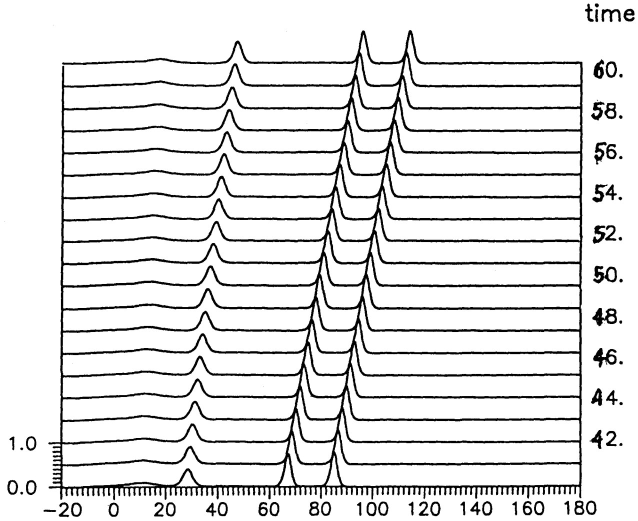
The results of the present Subsection allow as to claim the coherent structures of dissipation modified KdVE as solitons. We extend thus the Zabusky and Kruskal's definition to waves that are not of permanent shape. There is no crime in doing this, because even the proton is known to be unstable particle that eventually decays. The only problem is the time scale, because physically speaking there is no "infinite" time for which the waves (or particles) could attain the truly permanent shapes. What we claim is that in times lesser than "practical infinity" (the latter represented by the inverse of the small parameter ε^{-1}), the interactions of solitary waves (or wave crests) is almost solitonic with tolerable degree of inelasticity. The only difference with Zabusky and Kruskal is that here a recurrence of the initial state is impossible because of dissipative nature of the equation itself. In very many experimental situations (e.g., the Linde experiments) the experimental time is short enough and what is observed is transients of the above discussed type whose "aging" is slow enough to allow calling them solitons.

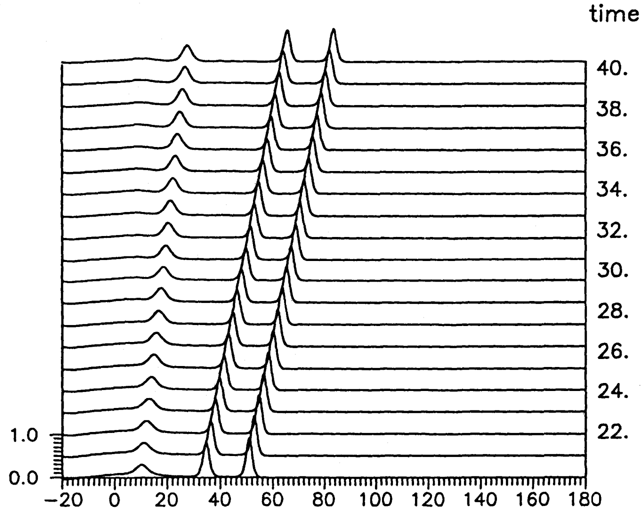
3.3. Collisions of Essentially Dissipative Solitons

In this Subsection we turn to true permanent solutions of KdV–KSVE. While the case with dissipation being relatively small disturbance is pertinent to the specific physical condition in thin liquid layers, the quest for permanent solutions and investigation of their collisions are of primary importance for extending the soliton paradigm into the realms of dissipation–dominated systems.

The permanent solitary waves of KSE are possible when the contributions of dif-

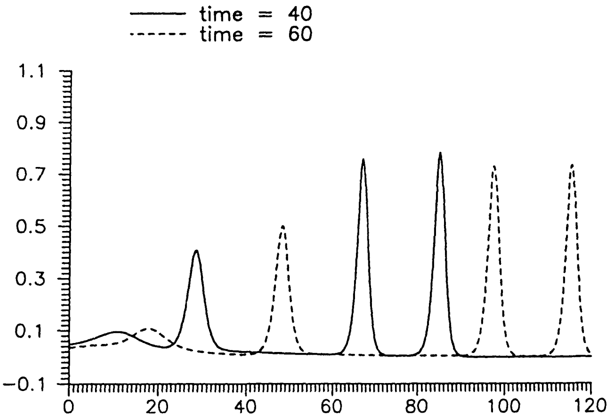
[§]One should be aware that words like "tighter", "looser", etc., only have meaning relatively to the scale of the solution. We appeal here to their intuitive meaning without going into quantitative details.





b) Evolution of the bound state;

Figure 6: Continuation



c) comparison of the solution for two different long times.

Figure 6: Continuation

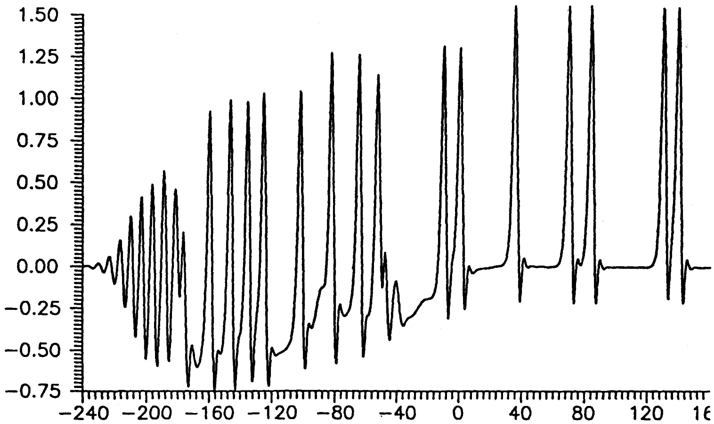
ferent terms (production, dissipation, nonlinearity) are in balance. If the dissipation enters the picture multiplied by a small parameter, then the permanent solution is necessarily a **short–wave** one, i.e. the permanent solutions of the intrinsically dissipative system under consideration are never long ones. Or vice versa – on the class of true permanent solutions, the KdV–KSVE (6) is intrinsically dissipative, rather than a dissipation modified essentially elastic system.

Let us now concentrate in the present paper mostly on the effects contained in the original KS model and in the additional dispersion (proportional to α_3). For this reason we concern ourselves in what follows with the following values of the parameters:

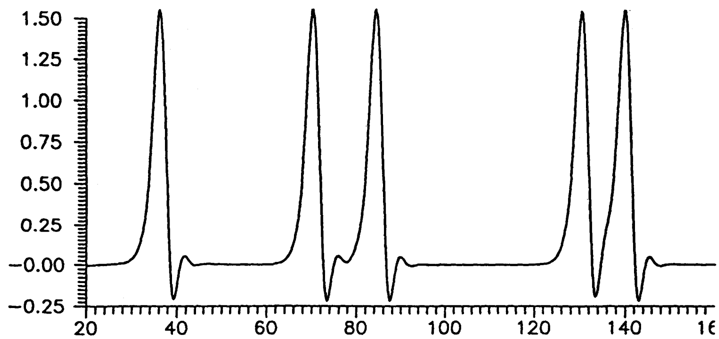
$$\alpha_1 = 3 \; , \qquad \alpha_2 = \alpha_4 = 1 \; , \qquad \alpha_5 = 0 \; . \tag{24}$$

We begin with the (α_3) interaction of two humps for pure KSE case. A hump of permanent form is necessarily a homoclinic solution of the QDE to which reduces the evolution equation in the moving frame.

The problem of identification of a homoclinics is inverse one and needs special treatment. It turned out that a hump shape propagating to the right exists in KSE only for a single value of phase velocity $c = 1.216$[21,82] although in certain papers[11,18,42], the possibility that the spectrum could be continuous was not excluded. In[15] is provided an explanation of this purely numerical effect. As it turns out, heteroclinics does exist for continuous spectrum of phase velocities and has exactly the same undulate forerunner as the homoclinics. Then if one matches a exponentially decaying towards $-\infty$ tail to an arbitrary heteroclinics, one obtains a homoclinic shape which satisfies the equation everywhere, save the point of matching where only the second derivative is discontinuous. Then it is clear, that such a shape would give a very small residual if introduced into the equation (for the case $c = 1$ in Figs. 6 the integral of the



a) the appearance of the solution for $t = 30$;



b) the permanent shapes in the region $x > 0$ for $t = 40$.

Figure 7: Overtaking interactions of a *sech* with $c = 10$ and $c = 2$ in KDV–KSVE for $\varepsilon = \alpha_2 = \alpha_4 = 2.$ and small initial separation:

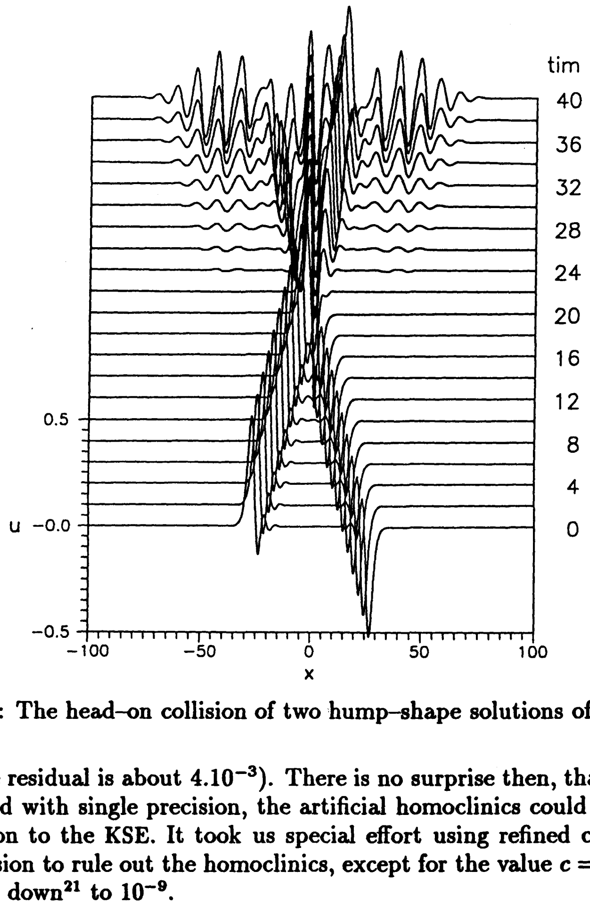


Figure 8: The head–on collision of two hump–shape solutions of KSE, eq.(3)

square of the residual is about 4.10^{-3}). There is no surprise then, that if calculations are conducted with single precision, the artificial homoclinics could be mistaken for a real solution to the KSE. It took us special effort using refined calculations with double precision to rule out the homoclinics, except for the value $c = 1.216$ when the residual goes down[21] to 10^{-9}.

Due to the symmetry of the problem the hump shape (homoclinics) exists also for $c = -1.216$, but with a negative amplitude. Unlike the *sech* solitons of KdVE it exhibits a wavy forerunner[21]. Since the magnitude of phase velocity is unique, then two coherent structures with phase velocities of the same sign could never collide since they can not catch up with each other. The only possible interaction is the ***head–on*** collision of two shapes with opposite phase velocities.

The result is depicted in Fig. 8 where the evolution of instability is clearly seen. The two humps collide and eventually form a single structure (see $t = 18, 20$). On this new shape, the conservation of energy is not maintained. Rather, the balance law provides for the evolution of the total energy, because the production and dissipation are not more in equilibrium, although they were equilibrated for each of the initial hump shapes and for their superposition when separated enough. After kicked out of equilibrium the system cannot return and the two humps do not re-emerge after the collision. Rather a new solution is formed (see Fig. 8, $t \geq 28$) for which the production of energy slightly exceeds the dissipation and hence the energy increases with time. We have already mentioned that the production (connected to the second derivative of the shape) could only exceed the dissipation (related to the fourth derivative) if the wave length is long enough. That is exactly what happens in the sequence of Fig. 8. It is clearly seen that Fig. 8 represents a slowly evolving signal which is in not a spurious numerical artifact.

The fact that initial shapes disappear after the interaction is enough to discard the humps of KSE as solitons, not to speak about the energy not being conserved in the course of collision. This does not, however, mean that a solitonic (in the sense of "particle–like") behaviour is strictly impossible in intrinsically dissipative systems. The problem with homoclinics of KSE is that its tail is unstable and as a result the shape is easily destroyed. The other candidates for soliton title are the heteroclinic solutions (kinks) for which both the tails and forerunners are undulate and stable (at least - metastable). One can expect then, that the distortion of a heteroclinic shape due to the interaction with another one, should not be so dangerous and would not switch the instability mechanism. That is why, it is to be expected that the heteroclinic shapes should persist even after a collision.

The first analytical kink solution (a combination of hyperbolic tangents) was obtained as early as in the original work of Kuramoto & Tsuzuki[48]. The analytical solution was found to exist only for a single value of the phase velocity. For the more general equation KdV–KSVE the same kind of solution was found in[47,26]. Our previous numerical work[15] based on the Method of Variational Imbedding had shown that various kink solutions do exist for all phase velocities $c > 0.3$. It is interesting to note that for intermediate values of phase velocity even more than one kink solution appears. When $c < 0.3$, the limit of linear waves is approached and no localized solutions of kink type are found numerically[15].

We investigated the collisions of kink solutions composing the initial condition of two kinks: the first (the left) between the levels $u_{left} = u_{-\infty}$ and $u_{right} = u_m$; and the second – between $u_{left} = u_m$ and $u_{right} = u_{+\infty}$. The latter means that for KdV–KSVE we can investigate the dynamics and interaction of kinks in a similar manner as it is usually done for the humps (*seches* of KdV. This is also clear physically because in a dissipative system in order to have a sustained structure one needs to pump energy. For the kinks the energy comes from the difference between the levels behind and ahead the wave. The dissipated in the shock energy is compensated by the difference of the levels of the solution at left and right infinity ($u_{-\infty}$ and $u_{+\infty}$), respectively and it is natural that for each value of the difference between levels a respective permanent kink–shape is formed. In this respect the KdV–KSVE is similar to Burgers equation where shocks are formed between two stationary states. Some comments on this point are due further on.

Details on numerical investigation are given in[16]. Here we present in Fig. 9 a typical scenario of overtaking collision. The faster kink catches up with the slower one and after a short period of collision they form a single kink proceeding with lesser

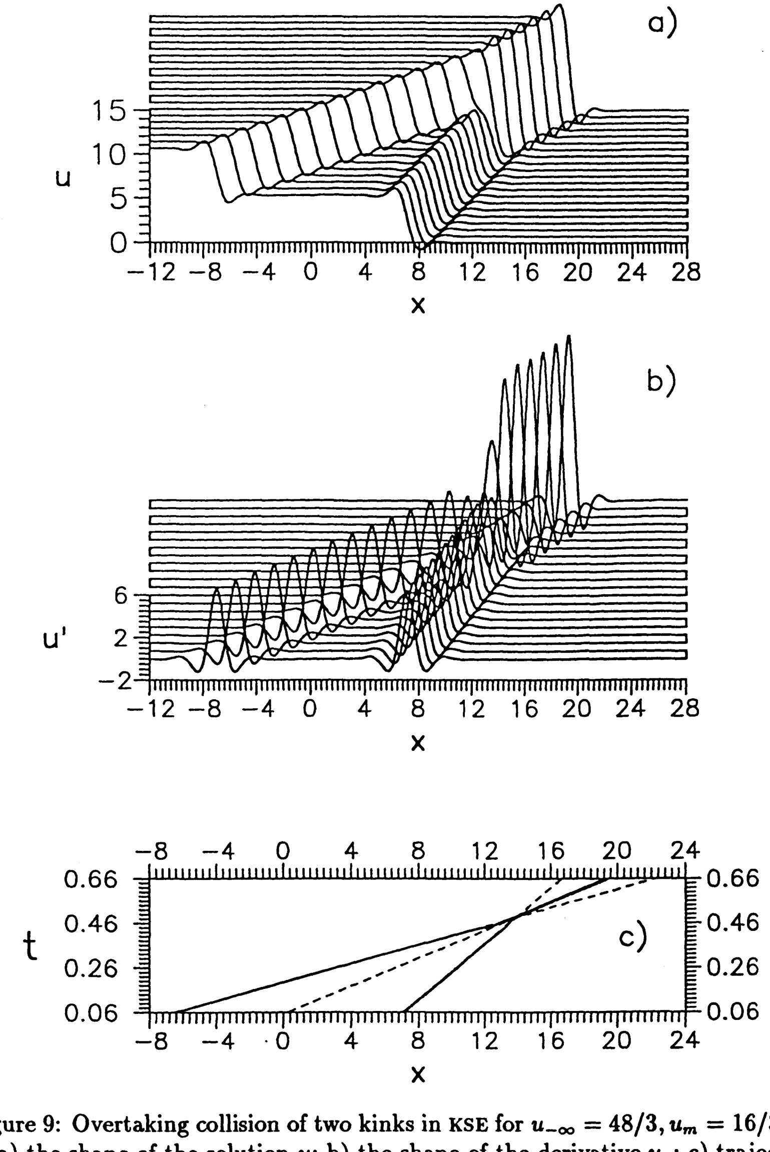


Figure 9: Overtaking collision of two kinks in KSE for $u_{-\infty} = 48/3, u_m = 16/3, u_{+\infty} = 0$: a) the shape of the solution u; b) the shape of the derivative u_x; c) trajectories of the "centers" of solitons.

phase velocity but with the same *momentum* (12) and a *mass* equal to the sum of masses of the two original kinks. The kink on the left are faster since it is between two higher levels (see (10) for the expression of the phase velocity). Even when the left kink has a lesser amplitude it still goes faster, because it is superimposed over the level of the right kink.

In Figs. 9–c are shown the trajectories of the "centers" of the coherent structures (kinks). A center of a kink is defined here as the point where $u = (u_{left} + u_{right})/2$. If the center is defined as the maximum of slope (maximum of first spatial derivative), the results differ insignificantly. According to this definition after the two kinks fuse to form a single one, then two different points are referred to as centers. Although very close, the trajectories of these two points are different and this is seen in the figures as two parallel lines that are very close to each other.

Although the interaction is completely inelastic, one can see that what can be conserved is indeed conserved and the trajectories exhibit no phase shift. The notion of phase shift, however, needs some clarification in this case. In lowermost figures are presented the trajectories of the kinks. In addition, the trajectories (dashed lines) of non-interacting kinks are also added. The dashed line in the middle of corresponding figure represents the estimate for the trajectory of a single kink with *mass* that is the sum of the two initial *masses*) and phase velocity defined from the total *momentum* when that said "compound" particle commences its motion from the position of the "mass center" of the system of two initial kinks. One sees that after the composite coherent structure (the composite *particle*) is formed in the numerical simulations, it proceeds exactly (within the error of the scheme) alongside the projected trajectory of the single *particle*-kink containing the total *mass* of the system. In this sense we claim that there is no phase shift in KSE. It is important to note here that the total pseudoenergy is not conserved during the collision and the final state has the same *momentum* but different *pseudoenergy*. In fact part of the initial *pseudoenergy* of the system of two kinks has been transformed into "internal energy" of the deformation suffered by the wave profile. The picture is the same if we consider the head–on collision of kinks.

We may speak of "completely inelastic interaction". In order to especially stress out the completely inelastic nature of the interactions among the coherent structures of the intrinsically dissipative system as KdV–KSVE we prefer to use the coinage clayons stemming from the notion that after the collision two solitary waves of KdV–KSVE stick to each other as two clay balls.

The results for the shape of the bore (Fig. 10) are of separate interest out of the context of the interactions. Localized solutions of type of stationary propagating bore (hydraulic jump) do not exist for the KDVE[89], because the undulate tail does not decay at $-\infty$. As shown numerically by Peregrine[64], the undular bore is not a stationary solution, rather its wavy tail spans ever larger intervals with time. On can indeed localize the bore only through adding dissipation. It was done by Johnson[37] who discussed the applicability of Korteweg–de Vries–Burgers equation KdVBE in which dissipation is represented by the usual second–order spatial derivative. He had shown, however, that there was a threshold for the amplitude of longitudinal velocity

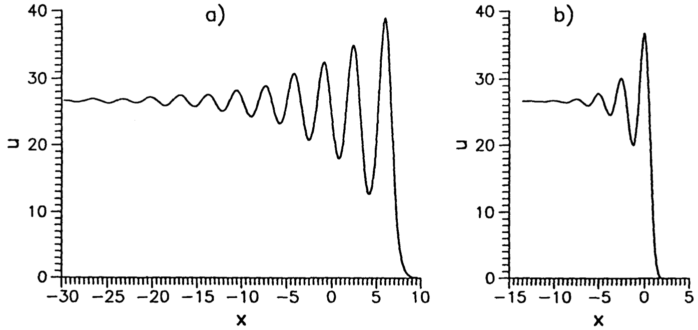


Figure 10: The role of dispersion in forming the shape of the kink in KDV–KSVE eq.(6): a) $\alpha_3 = 10$; b) $\alpha_3 = 5$.

beyond which the sign of the second derivative changes to the improper one (which is in fact the sign in KSE). In this instance, our bores Fig. 10 are of radically different nature than the bores of[37], the former being damped by the fourth derivative and excited by the term which played a damping role in[37]. However, from the point of view of the out–of–context paradigmatic approach taken here, the two cases are rather similar, because *first*: the presence of dissipation is what makes a stationary bore possible and *second*: an increase of dissipation decreases the undulations of the bore (according to our view, reduces the time allowed for the practical permanence of the bore). Vice versa – the increase of the dispersion parameter (Fig. 10) increases the undulations of the bore. The only difference here is that because of the presence of the production term, the bore is never monotonic in the KdV–KSVE even for vanishing value of the dispersion parameter $\alpha_3 = 0$.

4. Concluding Remarks

Three decades ago Zabusky and Kruskal[91,90] pioneered the work on solitons by exploring the behaviour of solutions of KdVE, which is a mathematical object containing an appropriate (local) balance between nonlinearity and dispersion. This balance allows for traveling permanent localized structures (solitary waves and crests of periodic wave trains/ series of solitons) that behave upon collisions as stable particles. Hence the coinage *soliton* for designating solitary wave–particle. The Zabusky–Kruskal's and FPU's precursor computer experiments were of those cases where numerical computations revealed new and unexpected results. The KdVE was later found not to be an isolated curiosity and a new scientific field was uncovered. Their physical problem involved no dissipation as it referred to a collisionless plasma kinetic model.

In the present paper we have taken up Zabusky and Kruskal's problem, acknowledging dissipation as it arises in the description of some physical problems, e.g., surface waves or internal waves in driven systems which obey the KdV–KSV eq.(6).

First we have treated the case of physical relevance when the coefficient of KS part of the equation is small, i.e., when dissipation slightly modifies with a smallness parameter ε the KdVE. For this case we have shown that:

1. The (nearly) solitonic behaviour is preserved for times smaller than the characteristic time defined by the inverse of the small parameter ε. There is a difference: the KdVE *seches* start "aging" when introduced into KdV–KSVE and eventually assume the permanent shapes that are solutions of KdV–KSVE.

2. Selection of the phase velocity and a discrete spectrum is observed due to the production–dissipation mechanism which although an ε–small disturbance, destroys the integrability of the system. Each initial shape eventually ends up in one of the permanent shapes propagating with one of the phase velocities (terminal velocity) from the discrete spectrum.

3. If the KdVE–*seches* are initially close enough, they interact (almost) as solitons. After the collision they transform into permanent shapes propagating with the terminal velocity and form a bound state, the latter propagating as a single structure. If initially they are not close enough, they evolve into the permanent shape before they reach each other and form the bound state without passing through each other.

4. Increasing the value of the smallness parameter ε controlling the KS part exaggerates the described effects and a threshold is reached past which the solution goes chaotic.

We have also considered the dissipation–modified KdVE (6) as a mathematical object worth exploring in itself, i.e., as a prototype of non–integrable model problem with no conservation laws, but rather with input–dissipation energy balance. In doing so we depart from underlying specific physics, but try to uncover possibly new phenomena of some universality and hopefully genuine of dissipative nonlinear wave equations.

We have discovered that:

5. The production–dissipation energy balance is capable of sustaining permanent shapes just as the nonlinearity–dispersion balance does in integrable systems. The permanent hump shapes of KSE are represented by a single homoclinic solution with dimensionless phase velocity $c = \pm 1.216$. The more interesting is the class of heteroclinic permanent shapes (bores /kinks /hydraulic jumps) which are found to exist for $c > 0.3$.

6. Upon head–on collision the hump shapes do not recover and yield to an ever–evolving wave pattern which is not of permanent shape. In this sense the hump does not seem to qualify for a soliton.

7. The interaction of kinks produces a permanent shape of the same class (kinks), thus exhibiting particle–like behaviour. However, the interaction here is *completely inelastic* in the sense that after the collision a larger kink is formed from the two initial kinks and this new wave–particle proceeds with phase velocity defined by the conservation of *pseudomomentum*. This justifies calling the kinks "inelastic solitons" (or "clayons" since they behave as two clay balls).

8. The soliton/clayon properties of kinks are not disproved by the presence of dispersion. Simply, the shapes become more undulate, but the interaction remains completely inelastic even for dispersion coefficients 20 times larger than the production–dissipation coefficient ε. This means that in the class of permanent kink solutions the KdV–KSVE is essentially dissipative (parabolic).

9. The assertions 5–8 allows us to draw the conclusion that the notion of soliton (wave–particle) can be extended to the essentially dissipative systems, but the wave–particles collide in a completely inelastic manner forming upon collision a larger particle that carries the total momentum of the system.

We used fully implicit four–stage–in–time difference scheme with Newton's quasi-linearization of the nonlinear terms. The practical stability of the scheme is virtually unlimited which allowed us to go after the very–long time evolution of the solution.

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5. References

1. G. B. Airy, Tides and waves, *Encyclopaedia Metropolitana*, **5**, No392, (1845) 241–392.
2. H. Bazin, Model Equation for Long Waves in Nonlinear Dispersive Systems, *Mémoires Divey Savants à l'Académie des Sciences*, **19** (1865) 495.
3. G. Benettin, Ordered and chaotic motions in dynamical systems with many degrees of freedom. *J. Maths & Phys.*, **45** (1966) 150–155; In: *Molecular–Dynamics Simulation of Statistical–Mechanical Systems*, Eds. G. Ciccotti and W. G. Hoover, (North–Holland, Amsterdam, 1986) pp.232–252.
4. D. J. Benney, Long waves in liquid film, *J. Maths & Phys.*, **45** (1966) 150–155.
5. J. V. Boussinesq, Théorie de l'intumescence liquide appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire. *Comp. Rend. Hebd. des Seances de l'Acad. des Sci.*, **72** (1871) 755–759.
6. J. V. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horisontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *Journal de Mathématiques Pures et Appliquées*, **17** (1872) 55–108.

7. J. V. Boussinesq, Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal, *Comp. Rend. Hebd. des Seances de l'Acad. des Sci.*, **73** (1871) 256–260.
8. F. K. Browand and P. D. Weidman, Large scales in the developing mixing layer. *J. Fluid Mech.*, **76** (1976) 127–144.
9. R. K. Bullough, 'The Wave' 'Par Excellence', the solitary progressive great wave of equilibrium in the fluid: an early history of the solitary wave, In: *Solitons. Introduction and Applications*, Ed. M. Lakshmanan, (Springer, N. Y., 1988) 7–42.
10. D. Campbell, An introduction to nonlinear phenomena. In: *Lectures in the Sciences of Complexity*, Ed. D. L. Stein, (Addison–Wesley, Redwood City, 1989) pp.3–54.
11. H.–C. Chang, Evolution of nonlinear waves on vertically falling films – a normal form analysis, *Chem. Engn. Sci.*, **42** (1987) 515–533.
12. H.–C. Chang, E. A. Demekhin, and D. I. Kopelevich, Laminarizing effect of dispersion in an active–dissipative nonlinear medium. *Physica*, **D 63** (1993) 299–320.
13. H.–C. Chang, E. A. Demekhin, and D. I. Kopelevich, Nonlinear evolution of waves on a vertically falling film, *J. Fluid Mech.*, **250** (1993) 433–480.
14. C. I. Christov, Flows with coherent structures: application of random point functions and stochastic functional series, In: *Continuum Models and Discrete Systems, vol. 1*, Ed. G. A. Maugin, (Longman, London, 1990), pp.232–252.
15. C. I. Christov, Localized solutions for fluid interfaces via method of variational imbedding, *These Proceedings.*
16. C. I. Christov, Numerical investigation of the long–time evolution and interaction of localized waves, *These Proceedings.*
17. C. I. Christov, Stochastic régimes for Kuramoto–Sivashinsky equation. I – Random point approximation, *Ann. Univ. Sofia, Fac. Math. & Inform.*, **81** (1987) 121–134.
18. C. I. Christov and V. P. Nartov, Application of the method of variational imbedding for calculating shapes of solitons on falling thin films, In *Proc. Conf. on Centenary of Academician L. Tchakalov, Samokov, Bulgaria, 1986*, pp.135–142.
19. C. I. Christov and V. P. Nartov, *Random Point Functions and Large–Scale Turbulence.* (Nauka,Siberian Branch, Novosibirsk, 1992). In Russian.
20. C. I. Christov and M. G. Velarde, Inelastic interaction of Boussinesq solitons, *Int. J. Bifurcation & Chaos*, **5** (1994) (to appear).
21. C. I. Christov and M. G. Velarde, On localized solutions of an equation governing Bénard–Marangoni convection, *Appl. Math. Modelling*, **17** (1993) 311–320.
22. M. C. Cross and P. C. Hohenberg, Pattern formation outside of equilibrium, *Rev. Mod. Phys.*, **65** (1993) 851–1112.
23. J. W. Dally and S. C. Stephen, Jr., Characteristics of the Solitary Wave, *Trans. ASCE*, **118** (1953) 575–578.
24. P. G. Drazin and R. S. Johnson, *Solitons: an Introduction*, (Cambridge Uni-

versity Press, Cambridge, 1987).
25. C. Elphick, G. R. Ierley, O. Regev, and E. A. Spiegel, Interacting localized structures with galilean invariance, *Phys. Rev.*, **A44** (1991) 1110–1122.
26. P. G. Estevez and P. R. Gordoa, Generalized solutions of the perturbed KdV equation for convecting fluids, *J. Phys. A: Math. Gen.*, **19** (1992).
27. E. Fermi, J. R. Pasta, and S. M. Ulam, Studies of nonlinear problems, Los Alamos Nat. Lab. Report LA–1940 (1955), In *Collected Papers of Enrico Fermi*, (Univ. Chicago Press, Chicago, 1965), pp.978–988.
28. A. N. Garazo and M. G. Velarde, Dissipative Korteweg–de Vries description of Marangoni–Bénard oscillatory convection, *Phys. Fluids*, **A3** (1991) 2295–2300.
29. P. L. García–Ybarra, J. L. Castillo, and M. G. Velarde, Bénard–Marangoni convection with a deformable interface and poorly conducting boundaries, *Phys. Fluids*, **30** (1987) 2655–2661.
30. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Method for solving Korteweg–de Vries equation, *Phys. Rev. Lett.*, **19** (1967) 1095–1097.
31. B. Gjevik, Occurrence of finite–amplitude surface waves on falling liquid films, *Phys. Fluids*, **13** (1970) 1918–1925.
32. R. Grimshaw, Theory of solitary waves in shallow fluids, In: *Encyclopedia of Fluid Mechanics*, Ed. N. P. Cheremisinoff, (Gulf Publishing Co., Houston, 1986) 3–25.
33. G. M. Homsy, Model equations for wavy viscous film flow, *Lect. in Appl. Math.*, **15** (1974) 191–194.
34. J. M. Hyman and B. Nicolaenko, Coherence and chaos in Kuramoto–Velarde equation, In: *Directions in Partial Differential Equations*, Eds. M. G. Grandall, P. H. Rabinovitz, and R. E. L. Turner, (Academic Press, 1987), pp.89–111.
35. J. M. Hyman and B. Nicolaenko, The Kuramoto–Sivashinsky equation: a bridge between PDE's and dynamical systems, *Physica*, **D 18** (1986) 113–126.
36. B. Janiaud, A. Pumir, D. Bensimon, V. Croquette, H. Richter, and L. Kramer, The Eckhaus instability for travelling waves, *Physica*, **D 55** (1992) 269.
37. R. S. Johnson, Shallow water waves on a viscous fluid. The undular bore, *Phys. Fluids*, **15** (1972) 1639–1699.
38. S. W. Joo, S. H. Davis, and S. G. Bankoff, Long–wave instabilities of heated falling films: two–dimensional theory of uniform layers, *J. Fluid Mech.*, **230** (1991) 117–146.
39. P. L. Kapitza, Wave flow in thin layer of viscous liquid, *Zh. Exp. Theor. Fiz.*, **18** (1948) 3–18. (in Russian).
40. P. L. Kapitza and S. P. Kapitza, Wave flow in thin layer of a viscous liquid – III. Experimental study of undulatory flow conditions, *Zh. Exp. Theor. Fiz.*, **19** (1948) 105. (in Russian).
41. P. L. Kapitza and S. P. Kapitza, Wave flow of thin layers of a viscous fluid, In: *Collected Papers of P. L. Kapitza, vol.2*, Ed. D. ter Haar, (Pergamon, Oxford, 1965) pp.690–709.

42. T. Kawahara and S. Toh, Pulse interactions in an unstable dissipative–dispersion nonlinear system, *Phys. Fluids*, **31**, (1988) 2103–2111.
43. G. H. Keulegan and G. W. Patterson, Mathematical theory of irrotational translation waves, *J. Research NBS*, **24** (1940) 47–101.
44. B. K. Kopbosynov and V. V. Pukhnatchev, Thermocapillary motion in thin liquid layer, In *Gidromekhanika i Protzessy Perenosa v Nevesomosti*, (Sverdlovsk, 1983). (in Russian).
45. D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Phil. Mag.*, **ser.5, 39** (1895) 422–443.
46. M. V. G. Krishna and S. P. Lin, Non–linear stability of a viscous film with respect to three–dimensional side-band disturbances, *Phys. Fluids*, **20** (1977) 1039–1044.
47. N. A. Kudryashev, Exact soliton solutions for a generalized equation of evolution for the wave dynamics, *Prikladnaya Math. i Mekhanika*, **52** (1988) 465–470. (in Russian).
48. Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Progr. Theor. Phys.*, **55** (1976) 356–369.
49. H. Lamb. *Hydrodynamics, 6th Edition 1879*, (Dover Reprints, New York, 1945).
50. P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.*, **21** (1968) 467–490.
51. S. P. Lin, Finite–amplitude side–band instability of a viscous film, *J. Fluid Mech.*, **63** (1974) 417–429.
52. S. P. Lin and C. Y. Yang, Modeling wavy film flows, In.*Encyclopedia of Fluid Mechanics, vol. 1*, Ed. N. P. Cheremisinoff, (Gulf Publishing Co., Houston, 1985), pp.931–951.
53. H. Linde, X.–L. Chu, and M. G. Velarde, Oblique and head–on collisions of solitary waves in Marangoni–Bénard convection, *Phys. Fluids*, **A5** (1993) 1068–1070.
54. H. Linde, X.–L. Chu, M. G. Velarde, and W. Waldheim, Wall reflections of solitary waves in Marangoni–Bénard convection, *Phys. Fluids*, **A5** (1993) 3162–3166.
55. S.–Y. Lou, G.–X. Huang, and X.–Y. Ruan, Exact solitary waves in a convecting fluid, *J. Phys. A: Math. Gen.*, **24** (1991) L587–L590.
56. T. Maxworthy, Experiments on collisions between solitary waves, *J. Fluid Mech.*, **76** (1976) 177–190.
57. J. McCowan, On the Solitary Wave, *The London, Edinburgh, and Dublin Phil. Mag. and Journal of Sci.*, **32(5)** (1895) 45.
58. C. Nakaya, Long waves on a thin fluid layer flowing down an inclined plane, *Phys. Fluids*, **18** (1975) 1407–1412.
59. A. A. Nepomnyashchy and M. G. Velarde, A three–dimensional description of solitary waves and their interaction in Marangoni–Bénard layers, *Phys. Fluids*,

6 (1994) 187.
60. C. Normad, Y. Pomeau, and M. G. Velarde, Convective instability. a physicist's approach, *Rev. Mod. Phys.*, **49** (1977) 581–601.
61. A. Oron and Ph. Rosenau, Formation of patterns induced by thermocapillarity and gravity, *J. Phys. II (Paris)*, **2** (1992) 131–146.
62. D. N. Payton III, M. Rich, and W. M. Visscher, Lattice thermal conductivity in disordered harmonic and anharmonic crystal models, *Phys. Rev.*, **160** (1967) 706–711.
63. D. N. Payton III and W. M. Visscher, Dynamics of distorted harmonic lattices II. The normal modes of isotropically disordered binary lattices, *Phys. Rev.*, **156** (1967) 1032–1038.
64. D. H. Peregrine, Calculations of the development of an undular bore, *J. Fluid Mech.*, **25** (1966) 321–330.
65. A. Pumir, P. Manneville, and Y. Pomeau, On solitary waves running down an inclined plane, *J. Fluid Mech.*, **135** (1983) 27.
66. Lord. Rayleigh, On waves, *Phil. Mag.*, **1** (1876), 257–279.
67. A. Ye. Rednikov, M. G. Velarde, Yu. S. Ryazantsev, and A. A. Nepomnyashchy. Bottom friction and nonlinear dissipative effects on wave trains and solitary waves in Marangoni–Bénard layer. *J. Fluid Mech.*, (1994) (submitted).
68. R. Rich, M. Visscher, and D. N. Payton, Thermal conductivity of two-dimensional two-branch lattice, *Phys. Rev. A*, **4** (1971) 1682–1683.
69. Ph. Rosenau and A. Oron, Bounded and unbounded features of the Benney equation, *Phys. Fluids*, **A4** (1992) 1102–11.
70. J. S. Russell, On waves, In: *Report of 14th Meeting (1844) of the British Association for the Advancement of Science*, (York, 1845), pp.311–390.
71. J. Sander and K. Hutter, On the development of the theory of the solitary wave. a historical essay, *Acta Mechanica*, **86** (1991) 111–152.
72. H. Segur, Who cares about integrability? *Physica*, **D 51** (1991) 343–359.
73. B. Sharbanov and C. I. Christov, Stochastic régimes for Kuramoto–Sivashinsky equation. II – Numerical simulation, *Ann. Univ. Sofia, Fac. Math. & Inform.*, **81** (1987) 135–144.
74. G. I. Sivashinsky and D. M. Michelson, On irregular wavy flow of a liquid film down a vertical plane, *Progr. Theor. Phys.*, **63** (1980) 2112–2114.
75. G. G. Stokes, On the Theory of Oscillatory Waves, *Cambr. Trans.*, **8** (1847) 441–473.
76. G. G. Stokes, On Waves, *Cambr. and Dub. Math. J.*, **IV** (1849) 219.
77. M. Toda, *Nonlinear Waves and Solitons*, (Kluwer, Dordrecht, 1989).
78. M. Toda, Studies on a nonlinear lattice, *Phys. Rep. (Phys. Lett. C)*, **18** (1974) 1.
79. M. Toda, *Theory of Nonlinear Lattice (2nd Ed.)*. (Springer–Verlag, N. Y., 1989).
80. M. Toda, Vibration of chain with nonlinear interaction, *J. Phys. Soc. Japan*, **22** (1967) 431–436.
81. M. Toda, Wave propagation in anharmonic lattices, *J. Phys. Soc. Japan*, **23**

(1967) 501–506.
82. O. Yu. Tzvelodub, Stationary Propagating Waves on a Thin Film Falling down a Vertical Wall, In: *Volnovie Protzessy v Dvukhfaznykh Sredakh*, Ed. V. E. Nakoryakov, (Novosibirsk, 1980), pp.47–63 (in Russian).
83. F. Ursell, The long-wave paradox in the theory of gravity waves, *Proc. Cambridge Phil. Soc.*, **49** (1953) 685–694.
84. F. Vander–Blij, Some details of the history of the Korteweg–de Vries equation, *Nieu Archief voor Wiskunde*, **26** (1978) 54–64.
85. M. G. Velarde and C. Normand, Convection, *Sci. Am.*, **243** (1980) 92–106.
86. M. G. Velarde (Ed.), *Physicochemical Hydrodynamics. Interfacial Phenomena.* (Plenum, New York, 1987).
87. M. G. Velarde, H. Linde, A. A. Nepomnyashchy and W. Waldhelm, Further evidence of solitonic behaviour in Bénard–Marangoni Convection, *These Proceedings.*
88. P. Weidman, H. Linde, and M. G. Velarde, Evidence for solitary wave behaviour in Marangoni–Bénard convection, *Phys. Fluids*, **A4** (1992) 921–926.
89. G. B. Whitham. *Linear and nonlinear waves*, (J. Wiley, N.Y., 1974).
90. N. J. Zabusky, Nonlinear lattice dynamics and energy sharing, *J. Phys. Soc. Japan*, **26** (1969) 196–202.
91. N. J. Zabusky and M. D. Kruskal, Interaction of 'solitons' in collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.*, **15** (1965) 57–62.