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SOLITON-SUPPORTING MODEL FOR DISPERSIVE SHALI OW-WATER FLOWS

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1. Introduction

After John Scott-Russell discovered the "great wave" there were different attempts to explain its existence and to find its appropriate model. Boussinesq [5, 3, 4] introduced the fundamental idea of balance between the nonlinearity and dispersion and derived the first approximate expression for the dispersion in the case of weakly nonlinear long waves. We call this balance "Boussinesq Paradigm" together with the set of different Boussinesq equation that are derived under the said assumption. They are *generalized wave equations* offering the opportunity to investigate the generic features of wave systems, such as head-on collisions of localized structures (solitary waves/quasi-particles). Boussinesq equations are not always fully integrable. As a rule they possess at least three conservation/balance laws – for mass, energy and momentum. Since not all of them are fully integrable, the problem arises of how to treat them numerically. In previous papers of the author [6, 9, 7] the way to construct conservative schemes for the Boussinesq Paradigm was outlined and their efficiency was demonstrated. In this paper we derive somewhat more general form (preserving the Galilean invariance) of the dispersive shallow water equations which is energy consistent.

2. Dispersive Shallow Water System (DSWS)

In this section we revisit Boussinesq's derivation with the purpose of improving it in the sense of conservation of the energy.

2.1. Posing the Problem

Consider the $2D$ inviscid flow in a thin layer with free surface. We restrict the derivations to the case when the shape function $h(x,t)$ of the free surface is single-valued, i.e., there is no breaking of the waves. The motion is governed by the Laplace equation for the potential Φ of the flow in the bulk

Let H be the scale for the vertical spatial coordinate and L (the yet undefined wave length) for the horizontal one we introduce dimensionless variables according to the

scheme

$$\Phi = UL\phi\,, \quad h = H\eta\,, \quad z = Hz'\,, \quad x = Lx'\,, \quad y = Ly'\,, \quad t = LU^{-1}t'\,,$$

where $U = \sqrt{gH}$ is the characteristic scale for the velocity. Henceforth, the primes will be omitted without fear of confusion. In terms of dimensionless variables the Laplace equation takes the form

$$\beta\Delta\phi + \frac{\partial^2\phi}{\partial z^2} = 0\,. \tag{2.1}$$

Here $\beta \equiv H^2L^{-2}$ is called dispersion parameter and it is a small quantity for length scales of motion long compared to the depth of the layer. The kinematic and dynamic conditions then become (the free surface in dimensionless form is $z = 1 + \eta$):

$$\frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta = \frac{1}{\beta}\frac{\partial\phi}{\partial z}\,, \tag{2.2}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2\beta}\left(\frac{\partial\phi}{\partial z}\right)^2 + \eta = 0\,. \tag{2.3}$$

The problem which remains is to identify the term containing the transverse derivative (z-derivative) of the potential at the free surface through some functions of the surface variables only. This can be done only approximately unless one uses the integral-equation formulation.

2.2. Boussinesq's Variables

Boussinesq expanded the solution of Laplace equation (2.1) into a power series with respect to β. Acknowledging the non-flux condition at the bottom of the layer it turned out to be a series with respect to the even powers of the coordinate z, namely

$$\phi(x,y,z,t) = \sum_0^\infty (-\beta\Delta)^m f(x,y,t)\frac{z^{2m}}{(2m)!}\,. \tag{2.4}$$

where $f(x,y,t) = \phi(x,y,z=0,t)$ is the unknown function representing the value of potential at the bottom of the layer. Then for the derivatives entering the surface conditions (2.2),(2.3) one has

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=1+\eta} = \sum_1^\infty (-\beta\Delta)^m f(x,y,t)\frac{(1+\eta)^{2m-1}}{(2m-1)!}\,, \tag{2.5}$$

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=1+\eta} = \sum_0^\infty (-\beta\Delta)^m \frac{\partial f(x,y,t)}{\partial t}\frac{(1+\eta)^{2m}}{(2m)!}\,, \tag{2.6}$$

$$\left.\nabla\phi\right|_{z=1+\eta} = \sum_0^\infty (-\beta\Delta)^m \nabla f(x,y,t)\frac{(1+\eta)^{2m}}{(2m)!}\,, \tag{2.7}$$

Introducing these expressions into the system governing the surface motion and neglecting the terms proportional to β^2 and the higher powers of β, one arrives at the

following approximate system containing the $1D$ variables η, f, only:

$$\frac{\partial \eta}{\partial t} + \left[\nabla f - \frac{\beta}{2}\nabla[(1+\eta)^2\Delta f]\right]\cdot\nabla\eta = -(1+\eta)\Delta f + \frac{\beta}{6}(1+\eta)^3\Delta^2 f\,, \quad (2.8)$$

$$\frac{\partial f}{\partial t} - \frac{\beta}{2}\frac{\partial}{\partial t}[(1+\eta)^2\Delta f] + \frac{1}{2}(\nabla f)^2 + \eta - \frac{\beta}{2}\nabla f\cdot\nabla\left[(1+\eta)^2\Delta f\right] + \frac{\beta}{2}[(1+\eta)\Delta f]^2 = 0\,, \quad (2.9)$$

Note that in the $2D$ case (no dependence on y) one has $\Delta \equiv \partial^2/\partial x^2$ and $\nabla \equiv \partial/\partial x$.

2.3. Asymptotic Simplifications

It is clear that when the dispersion parameter β is small then the dominating terms of order $O(1)$ represent the propagating of waves in a hyperbolic system. If now we admit existence of waves of amplitude of order $|\eta|, |f| \simeq O(1)$, then we have to keep within the said asymptotic order the nonlinear terms while at the same time the dispersive terms will be of order $O(\beta)$. Then the well known phenomenon of steepening of the surface waves takes place which is the reason why Airy [1] ruled out the existence of permanent waves, actually observed by Russell [13, 14]. The main idea of Boussinesq was to look for weakly nonlinear waves of amplitude $|\eta|, |f| \simeq O(\beta)$ when the leading order is $O(\beta)$ reducing to a hyperbolic equation for linear propagation of waves. In the next order $O(\beta^2)$ one has a balance of two small effects – nonlinearity and dispersion. This balance is quite fortunate a thing since it turns out to be valid at each point for the whole wave shape (the famous *sech* discovered by Boussinesq). This balance secures that the wave will propagate as a linear wave according to the equation on the leading asymptotic order and will remain unchanged if left alone (without collisions with other waves.

What we call "Boussinesq approximation" consists in assuming that $\eta = \beta\bar{\eta}, f = \beta\bar{f}$ and keeping the terms up to the order $O(\beta^2)$ including. Then

$$\begin{aligned}
\beta(1+\eta)^3\Delta f &= \beta^2\Delta\bar{f} + O(\beta^3) \\
\frac{\beta}{2}\nabla\left[(1+\eta)^2\Delta f\right]\cdot\nabla\eta \simeq \frac{\beta}{2}\nabla\left[(1+\eta)^2\Delta f\right]\cdot\nabla f &= O(\beta^3) \\
\frac{\beta}{2}\frac{\partial}{\partial t}\left[(1+\eta)^2\Delta f\right] &= \frac{\beta^2}{2}\frac{\partial\Delta\bar{f}}{\partial t} + O(\beta^3) \\
\frac{\beta}{2}\left[(1+\eta)^2\Delta f\right]^2 &= O(\beta^3)
\end{aligned}$$

and the $O(\beta^3)$ approximation of the system (after dividing by β) reads

$$\frac{\partial\bar{\eta}}{\partial t} + \beta\nabla\cdot\bar{\eta}\nabla\bar{f} = -\Delta\bar{f} + \frac{\beta}{6}\Delta^2\bar{f} + O(\beta^2)\,. \quad (2.10)$$

$$\frac{\partial\bar{f}}{\partial t} - \frac{\beta}{2}\frac{\partial\Delta\bar{f}}{\partial t} + \frac{\beta}{2}(\nabla\bar{f})^2 + \bar{\eta} = 0 + O(\beta^2)\,. \quad (2.11)$$

From now on the overbars will be omitted without fear of confusion.

This system has a major disadvantage in comparison with the original one because it does not bring about the conservation of energy. In this sense, it is not yet the desired system for numerical computations. In order to find the correct energy conserving form of the system we introduce the auxiliary variable

$$\chi = \eta - \frac{\beta}{2}\frac{\partial \Delta f}{\partial t}$$

Upon introducing the last relation in (2.10), (2.11) we get

$$\frac{\partial \chi}{\partial t} + \beta \nabla \cdot \chi \nabla f + \frac{\beta^2}{2}\nabla \cdot \left(\frac{\partial \Delta f}{\partial t}\nabla f\right) = -\Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}\frac{\partial^2 \Delta f}{\partial t^2},$$

$$\frac{\partial f}{\partial t} = -\frac{\beta}{2}(\nabla f)^2 - \chi.$$

Now it is clear that in the framework of weakly nonlinear approximation the term $-\frac{\beta^2}{2}\nabla \cdot \left(\frac{\partial \Delta f}{\partial t}\nabla f\right)$, can be neglected within the adopted asymptotic order $O(\beta^2)$. Then the system adopts the following simple form

$$\frac{\partial \chi}{\partial t} = -\beta \nabla \cdot \chi \nabla f - \Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}\frac{\partial^2 \Delta f}{\partial t^2}, \tag{2.12}$$

$$\frac{\partial f}{\partial t} = -\frac{\beta}{2}(\nabla f)^2 - \chi. \tag{2.13}$$

Upon multiplying the left-hand side of eq.(2.12) by the right-hand side of eq.(2.13) and the right-hand side of eq.(2.12) by the left-hand side of eq.(2.13), adding the results and integrating over the surface region D under consideration one gets the following energy balance law

$$E = \frac{1}{2}\int_D \left[\chi^2 + (1+\beta\chi)(\nabla f)^2 + \frac{\beta}{6}(\Delta f)^2 + \frac{\beta}{2}(\nabla f_t)^2\right] dx$$

$$\frac{dE}{dt} = \oint_{\partial D}\left[(1+\beta\chi)f_t\frac{\partial f}{\partial n} + \frac{\beta}{2}f_t\frac{\partial f_{tt}}{\partial n} + \frac{\beta}{2}f_t\frac{\partial \Delta f}{\partial n} - \frac{\beta}{2}\Delta f\frac{\partial f_t}{\partial n}\right] ds, \tag{2.14}$$

which allows us to call the system (2.12), (2.13) "Energy Consistent Boussinesq Paradigm". We believe that this is the system that fulfills the Boussinesq program without unnecessary deficiencies introduced by Boussinesq.

The boundary conditions that bring about the conservation of the total energy stem from the requirement that the right-hand side of (2.14) be equal to zero. There are three sets of conditions compatible with that requirement, namely

$$f_t = 0 \Longrightarrow f = f_b(x,y) \quad \text{and} \quad \frac{\partial f}{\partial n} = 0 \quad \text{for} \quad (x,y) \in \partial D \tag{2.15}$$

$$f_t = 0 \Longrightarrow f = f_b(x,y) \quad \text{and} \quad \Delta f = 0 \quad \text{for} \quad (x,y) \in \partial D \tag{2.16}$$

$$\frac{\partial f}{\partial n} = 0 \quad \text{and} \quad \frac{\partial \Delta f}{\partial n} = 0 \quad \text{for} \quad (x,y) \in \partial D \tag{2.17}$$

The third set of boundary conditions (2.17) are Neumann conditions and they specify function f up to an arbitrary function of time. The most suitable set of b.c. will be selected after considering the balance law for the wave momentum.

In terms of the original function η (meaning the elevation of the free surface) the energy consistent Boussinesq approximation reads

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \eta \nabla f - \frac{\beta^2}{2} \nabla \cdot \frac{\partial \Delta f}{\partial t} \nabla f = -\Delta f + \frac{\beta}{6} \Delta^2 f , \tag{2.18}$$

$$\frac{\partial f}{\partial t} - \frac{\beta}{2} \frac{\partial \Delta f}{\partial t} = -\frac{1}{2} (\nabla f)^2 - \eta , \tag{2.19}$$

with the energy functional readily derived from (2.14).

Here is to be mentioned that η and χ are implicit functions without b.c. for them hence there are no reasons to prefer one over the other. Hence we will never use the conservative system in original variables (2.18), (2.19).

2.5. Balance Law for the Pseudomomentum

Another balance (or conservation) law holds for the wave momentum (called sometimes *pseudomomentum* [10, 11]). In order to derive it for the system under consideration we make use of the following identities

$$\begin{aligned}
\Delta f \nabla f &= \nabla \cdot (\nabla f \nabla f) - \frac{1}{2} \nabla (\nabla f)^2 \\
\Delta f_t \nabla f_t &= \nabla \cdot (\nabla f_t \nabla f_t) - \frac{1}{2} \nabla (\nabla f_t)^2 \\
\Delta^2 f \nabla f &= \nabla \cdot (\nabla \Delta f \nabla f) - \frac{1}{2} \nabla (\nabla \Delta f \cdot \nabla f) \\
\nabla \cdot (\chi \nabla f \nabla f) &= \nabla f \nabla \cdot (\chi \nabla f) + \frac{\chi}{2} \nabla (\nabla f)^2
\end{aligned}$$

which are readily proved. Note that when there is no sign for scalar product "·" then an outer product (dyadics, polyadics, etc.) are meant, namely $\nabla f \nabla f$ is the dyadic tensor.

The pseudomomentum is defined as

$$\boldsymbol{P} = -\int_D \left[\chi \nabla f + \frac{\beta}{2} \frac{\partial \Delta f}{\partial t} \nabla f \right] \mathrm{d}x\mathrm{d}y . \tag{2.20}$$

and making use of the above equalities one obtains for the evolution of the total pseudomomentum the following

$$\begin{aligned}
-\frac{d\boldsymbol{P}}{dt} &= \int_D \left\{ -\left[\nabla \cdot (\nabla f \nabla f) - \frac{1}{2} \nabla (\nabla f)^2 \right] - \frac{1}{2} \nabla \chi^2 - \beta \left[\nabla f \nabla \cdot (\chi \nabla f) + \frac{\chi}{2} \nabla (\nabla f)^2 \right] \right. \\
&\quad \left. + \frac{\beta}{6} \left[\nabla \cdot (\nabla \Delta f \nabla f) - \frac{1}{2} \nabla (\nabla \Delta f \cdot \nabla f) \right] + \frac{\beta}{2} \Delta f_t \nabla f_t \right\} \mathrm{d}x\mathrm{d}y \\
&= \oint_{\partial D} \left[-\frac{\beta}{2} \left(\frac{\partial f_t}{\partial n} \nabla f_t - \frac{1}{2} (\nabla f_t)^2 \right) \boldsymbol{n} - \frac{1}{2} \chi^2 \boldsymbol{n} - \left(\frac{\partial f}{\partial n} \nabla f - \frac{1}{2} (\nabla f)^2 \boldsymbol{n} \right) \right.
\end{aligned}$$

$$+\frac{\beta}{6}\left(\frac{\partial\Delta f}{\partial n}\nabla f-\frac{1}{2}\nabla\Delta f\cdot\nabla f\right)\boldsymbol{n}+\beta\chi\nabla f\frac{\partial f}{\partial n}\Big]\,\mathrm{d}s\,.$$

Here it becomes clear that the most natural boundary conditions are (2.15) because most of the terms in the balance law of the pseudomomentum cancel reducing it to the following

$$\frac{d\boldsymbol{P}}{dt}=\oint_{\partial D}\left\{-\frac{\beta}{4}(\nabla f_t)^2-\frac{1}{4}(\nabla f)^2+\frac{\beta}{12}(\nabla\Delta f\cdot\nabla f)+\frac{1}{2}\chi^2\right\}\boldsymbol{n}\,\mathrm{d}s\,. \tag{2.21}$$

which transforms into a conservation law if in (2.15) one takes $f_b(x,y)=0$.

3. Alternative Formulation

Upon introducing (2.13) into (2.12) the function χ is readily excluded to obtain a single equation for the potential f, namely

$$f_{tt}+2\beta\nabla f\cdot\nabla f_t+\beta f_t\Delta f+\frac{3\beta^2}{2}(\nabla f)^2\Delta f-\Delta f+\frac{\beta}{6}\Delta^2 f-\frac{\beta}{2}\frac{\partial^2\Delta f}{\partial t^2}=0\,, \tag{3.1}$$

with Hamiltonian density

$$\mathcal{H}=\frac{1}{2}\left[f_t^2+(\nabla f)^2-\frac{\beta^2}{4}(\nabla f)^4+\frac{\beta}{6}(\Delta f)^2+\frac{\beta}{2}(\nabla f_t)^2\right] \tag{3.2}$$

Here becomes clear that the nonlinearity of the dynamic condition (2.13) is responsible for the cubic nonlinearity of eq.(3.1). It is of higher order in β, but it can not be neglected without destroying the Galilean invariance. Boussinesq [3, 4, 5] did neglect it and the equation he obtained lacked the said invariance (see Section 5).

The system described by eq.(3.1) can be re-interpreted in a field-theoretic framework (very much in the same vein as in our previous work concerning the 6th order Boussinesq equation [8]) by introducing also a Lagrangian density

$$L=\int_D\mathcal{L}\mathrm{d}x\mathrm{d}y\,,\quad \mathcal{L}=\mathcal{K}-\mathcal{W}\,,\quad \mathcal{H}=\mathcal{K}+\mathcal{W}\,,\quad P^w=-\int_D\nabla f\frac{\delta\mathcal{L}}{\delta f_t}dxdy\,, \tag{3.3}$$

$$\mathcal{K}=\frac{1}{2}f_t^2+\frac{\beta}{2}f_t(\nabla f)^2\,,\quad \mathcal{W}=\frac{1}{2}\left[(\nabla f)^2-\beta f_t(\nabla f)^2-\frac{\beta^2}{4}(\nabla f)^4-\frac{\beta}{3}(\Delta f)^2+\frac{\beta}{2}(\nabla f_t)^2\right].$$

In this expression in (3.3) one easily recognizes the pseudomomentum defined in (2.21), with an additional contribution probably due to the transport of the finite domain D when the Eulerian-Lagrangian passage is duly taken into account (see eq.(4.42) from [11]). The appropriate boundary conditions yield the balance of *wave momentum*.

4. One-Dimensional Version

Consider now the two-dimensional flow in which nothing depends on the coordinate y. Naturally it is 1D for the surface variables. Then the system (2.12), (2.13) reduces to

the following

$$\frac{\partial \chi}{\partial t} + \beta \frac{\partial}{\partial x}\left(\chi \frac{\partial f}{\partial x}\right) = -\frac{\partial^2 f}{\partial x^2} + \frac{\beta}{6}\frac{\partial^4 f}{\partial x^4} - \frac{\beta}{2}\frac{\partial^4 f}{\partial t^2 \partial x^2}, \tag{4.4}$$

$$\frac{\partial f}{\partial t} + \frac{\beta}{2}\left(\frac{\partial f}{\partial x}\right)^2 = -\chi, \tag{4.5}$$

Being reminded that f has the meaning of potential of the flow velocity on the bottom of the layer one can introduce new variables $u = f_x$ and $\chi = -q_x$. Upon integrating eq.(4.4) once and acknowledging the boundary conditions one obtains

$$\frac{\partial q}{\partial t} + \beta u \frac{\partial q}{\partial x} = -u + \frac{\beta}{6}\frac{\partial^2 u}{\partial x^2} - \frac{\beta}{2}\frac{\partial^2 u}{\partial t^2}, \tag{4.6}$$

$$\frac{\partial u}{\partial t} + \beta u \frac{\partial u}{\partial x} = -\frac{\partial^2 q}{\partial x^2}, \tag{4.7}$$

The region under consideration is a finite interval, say $x \in [-L_1, L_2]$. Then boundary conditions (2.15) are reduced to the following

$$u = 0, \quad q_x = 0, \qquad x = -L_1, L_2. \tag{4.8}$$

5. Boussinesq Paradigm Equation

Boussinesq attempted to describe the nearly quasi-stationary wave phenomena in the moving frame. Then some simplifications of the original system are possible which can be broken into two parts. The first step is to simplify the convective nonlinear terms neglecting the nonlinearity in (2.13) or the cubic term in (3.1). If one restricts oneself to an approximation valid only in the moving frame, then following Boussinesq [3, 4, 5] one can argue that the time derivatives can be approximated by the spatial ones for motions that evolve slowly in the coordinate frame moving to the right with unit velocity. Then the time derivative f_t is replaced by f_x in the quadratic nonlinear term, arriving thus at

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \frac{dU}{du} + \frac{\beta}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\beta}{6}\frac{\partial^2 u}{\partial x^2}\right], \quad U(u) \equiv \frac{\beta}{2}u^3 \tag{5.1}$$

which we called Boussinesq Paradigm Equation (BPE) and treated numerically by a fully implicit conservative scheme in [7]. It should be mentioned that the eq.(5.1) appears also in the theory of longitudinal (acoustic) vibrations of rods (see, e.g., [15]) and in continuum limit for lattices (see, e.g., [16, 17]).

The above simplification destroyed the Galilean invariance of the system. The second part of Boussinesq simplifications consisted in substituting $\partial^2/\partial x^2$ for $\partial^2/\partial t^2$ arriving at

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \frac{3\beta}{2}u^3 + \frac{\beta}{3}\frac{\partial^2 u}{\partial x^2}\right]. \tag{5.2}$$

which we call Boussinesq's Boussinesq Equation (BBE). It is is linearly instable and cannot be solved numerically.

Another option is to render the spatial derivative in the brackets to a temporal and to represent the dispersion as $\frac{\beta}{3}u_{tttxx}$. The equation with the mixed fourth-order derivative but without fourth spatial derivative is called in the literature Regularized Long-Wave Equation (RLWE) (see, [2, 18]) the very name suggesting that something had to be regularized in the thin-film equations. The fact is that the equation naturally appearing in Boussinesq's type of derivation (see, previous subsection) is well posed and curiously enough some effort is needed to "de-regularize" it and to make it incorrect. Note that RLWE is linearly stable [12, 2, 18]) but it is not fully integrable (just as the original DSWS system is not).

Eq.(5.1) can be rewritten as a system

$$u_t = q_{xx}\,, \quad q_t = u - \frac{dU}{du} + \beta_1 u_{tt} - \beta_2 u_{xx} \tag{5.3}$$

where $\beta_1, \beta_2 > 0$ are introduced for the sake of generality. Following [6, 9] we show that for the b.c. (4.8) one obtains

$$\frac{dM}{dt} = 0\,, \quad M = \int_{-L_1}^{L_2} u dx \tag{5.4}$$

$$\frac{dE}{dt} = 0\,, \quad E = \frac{1}{2}\int_{-L_1}^{L_2} \left[q_x^2 + u^2 - 2U(u) + \beta_1 u_t^2 + \beta_2 u_x^2\right] dx \tag{5.5}$$

$$\frac{dP}{dt} = \left[\frac{u^2}{2} - \left(u\frac{dU}{du} - U(u)\right) - \frac{\beta_1}{2}u_t^2 - \frac{\beta_2}{2}u_x^2\right]_{-L_1}^{L^2} = -\frac{\beta_2}{2}u_x^2\Big|_{-L_1}^{L^2}\,,$$

$$P = \int_{-L_1}^{L_2} u\,(q_x + \beta_1 u_{xt})\,dx = \int_{-L_1}^{L_2} (uq_x - \beta_1 u_t u_x)\,dx\,. \tag{5.6}$$

Thus we see that if the second step of Boussinesq simplifications is not performed and BPE is left in its original form it is preferable over RLWE because now the wave *mass* is also conserved alongside with the *energy*. In other words, the presence of the spatial fourth derivative requires boundary conditions whose satisfaction brings about the conservation of the wave mass.

6. The Solitary Wave of DSWS

Although more elaborate than any of the Boussinesq equations, the DSWS system (4.6), (4.7) does admit localized solution stationary in the moving frame $x - ct$. After some algebra we found a *sech* solution, namely

$$u = \frac{a\,\mathrm{sign}(c)}{\frac{|c|-1}{2} + \cosh^2[b(x-ct)]}\,, \quad a = \frac{c^2-1}{\beta}\,, \quad b = \sqrt{\frac{(c^2-1)}{2\beta(c^2-\frac{1}{3})}}\,, \tag{6.1}$$

where c is the phase speed or *celerity* of wave. It is clear that this solution exists in the same range as the BPE solitons (see, [7]): either for supersonic celerities $c > 1$ or for $c < \sqrt{1/3}$. Once again only the supersonic *sech*-es are of physical relevance because for small β the subsonic ones cannot be long-length waves. The fact that DSWS admits also a *sech* solution renders the Boussinesq simplifications unnecessary. The *sech*-like solution (6.1) is another candidate for the John Scott Russell's "Great (or Permanent) Wave".

For the Boussinesq Paradigm Equation with $U(u) = \beta u^3/2$ the *sech* solitons are given by (see [7]):

$$u = \frac{a}{\cosh^2[b(x - ct)]}, \tag{6.2}$$

where the coefficients a and b and the range for celerity are the same as in (6.1).

Comparing (6.2) and (6.1) reveals that the only difference is the term $\frac{1}{2}(|c| - 1)$ which means that in the limit of weakly-nonlinear case $|c - 1| \sim O(\beta) \ll 1$ they will virtually coincide. For all c the said difference is small in the "tails" of the solution. It may have some significance only near the origin of coordinate system where the smallest value of the denominator ($cosh \approx 1$) and then only for significantly supercritical c.

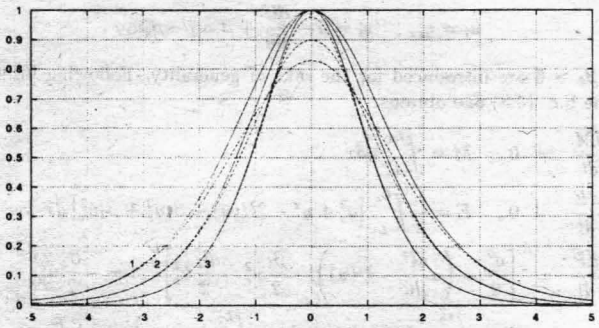

Figure 1: Comparison between the DSWS (dashed lines) and BPE (solid lines) *sech*-es for supercritical celerities $c = \sqrt{1 + \beta}$: $1 - \beta = 1$; $2 - \beta = 0.5$; $3 - \beta = 0.1$.

In order to keep within the long-wave approximation we have taken the supersonic celerities $c^2 = 1 + \beta$ and calculated the shapes of the solitary waves of DSWS and BPE. Fig. 1 shows the comparison between the two solutions. The DSW wave is always of smaller amplitude than the BPE one. For really small β the differences are quantitatively very small and it is hard to distinguish which one corresponds better to the experiment. A good case for comparison with the experiments of John Scott Russell could be $\beta = 0.2$.

7. Conclusions

The long-wave motion in shallow layer of inviscid liquid is considered. The Boussinesq derivation is applied for obtaining approximate expression for the dispersion when $\beta \equiv H^2/L^2 \ll 1$. Some terms of order $O(\beta^2)$ are neglected on the grounds that the conservation law for energy must hold for the approximate model (energy consistent derivation). The new system preserves also the Galilean invariance of the original problem which is its main advantage over the Boussinesq equations. Its Hamiltonian structure is derived and a new analytical solution of type of solitary wave is obtained.

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