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NONLINEAR DUALITY BETWEEN ELASTIC WAVES AND QUASI-PARTICLES IN MICROSTRUCTURED SOLIDS

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Abstract. The point mechanics of quasi-particles associated with solitary-wave solutions of nonexactly integrable systems is briefly discussed. This is illustrated by the case of so-called Kawahara solitons that are solutions of the generalized Boussinesq equations issued from the dynamics of microstructured crystal elasticity.

Key words: solitons, elasticity, conservation equations, duality, microstructure.

1. DUALITY AND POINT MECHANICS

Exactly integrable systems of *soliton theory* admit an infinite hierarchy of conservation laws (see, e.g., [1]). But many realistic physical systems of partial differential equations (PDEs) which exhibit *solitary waves* are in general *not* exactly integrable, a situation that prevails in deformable solids with a microstructure. Still, if they are based on a field-theoretic approach, such as nonlinear elasticity with strain gradients or models of coupled fields (elastic ferromagnets), they certainly admit a minimal set of *canonical balance laws* bearing a true physical significance (which is *not* the case of higher-order conservation laws in soliton theory). These can be used either to study the closeness of the dynamic behaviour exhibited by solitary waves to a pure *solitonic* one (how much deviation from conservation of basic canonical quantities is observed?) or as a practical means to evaluate the perturbations caused by additional terms in the systems in question. This point of view is close to that of *perturbed balance laws* in [2] and that of *collective coordinates* in [3].

The basic *canonical balance equations*, which pertain to the whole *wave solution*, consist essentially of the conservation of total *mass*, *energy*, and *wave momentum* or quantities that are interpreted as such. These quantities, defined in an appropriate and *consistent* manner, provide a *point mechanics* of the wave-like objects, hence a one-way *duality* between certain nonlinear wave processes and a quasi-particle viewpoint. A pure solitary wave behaviour (propagation without disturbance in shape, speed, and phase) corresponds to an *inertial motion* in that mechanics. A *perturbed* (*accelerated or decelerated*) *motion* corresponds to the presence of a *driving force* acting on the wave form.

The crux of the matter obviously consists in identifying the *point mechanics* that is associated with the initial system of *nonlinear dispersive* PDEs. It is appropriate to recall that, in one space dimension, a *point mechanics* is the datum of (i) a *rest mass* M_0, (ii) a definite relationship $P = \overline{P}(M_0, V)$, where P is the *momentum* of the quasi-particle and V is the velocity of the mass centre of the "wave object", and (iii) a relationship $E = \overline{E}(P, M_0)$, where E is the *energy* of the quasi-particle. The last two relations must be consistent. If τ is a time measured following the quasi-particle along its trajectory in space-time, then for an *inertial motion* we have the *"pseudo-Galilean"* equation of motion: $dP/d\tau = 0$, whence $V = \text{const.}$

In classical mechanics we know only of *Newtonian* and *Lorentzian–Einsteinian* point mechanics which indeed present characteristically different functional relationships $\overline{P}(M_0, V)$ and $\overline{E}(P, M_0)$, but an identical Galilean inertial equation of motion. In soliton theory, it is known that the (i) *kink solutions* of the *sine-Gordon* (SG) PDE are associated to quasi-particles having a Lorentzian–Einsteinian point mechanics (Fig. 1*b* shows $P(V)$ graphs with V noted c), and (ii) *dark* and *bright* soliton solutions of the *nonlinear Schrödinger* (NLS) *equation* are associated to quasi-particles having a Newtonian point mechanics (Fig. 1*a*). This shows that the *nonlinear* duality a priori looks somewhat unpredictable. Obviously, more *exotic* point mechanics may be associated with other systems of PDEs which, nonetheless, exhibit exact solitary-wave solutions. This is the case of *sine-Gordon–d'Alembert* (SGdA) systems obtained while studying the propagation of magnetoacoustic domain walls in elastic ferromagnets [4] and electroacoustic walls in elastic ferroelectrics equipped with a molecular group such as $NaNO_2$ [5]. Their quasi-particle mechanics deviates somewhat from the pure Lorentzian–Einsteinian mechanics.

As to the solitary-wave solutions of the *generalized-Zakharov* (GZ) *systems*, obtained while studying the propagation of surface solitary waves on a thin film glued on a nonlinear substrate [6], they exhibit a point mechanics as sketched in Fig. 1*c*. That is, they are quasi-Newtonian at small velocities, relativistic (Lorentzian–Einsteinian) at velocities close to, but less, than the characteristic velocity 1, while there exists a forbidden window between the values 1 and c^*, beyond which, for sufficiently high rest mass M_0 (interpreted as the number of surface phonons), the behaviour recurs to a quasi-Newtonian one for very high speeds, although it yields a catastrophe – related to the presence of a minimum in the momentum curve and facetiously called "perestroika" – for velocities in the intermediate supersonic range [7].

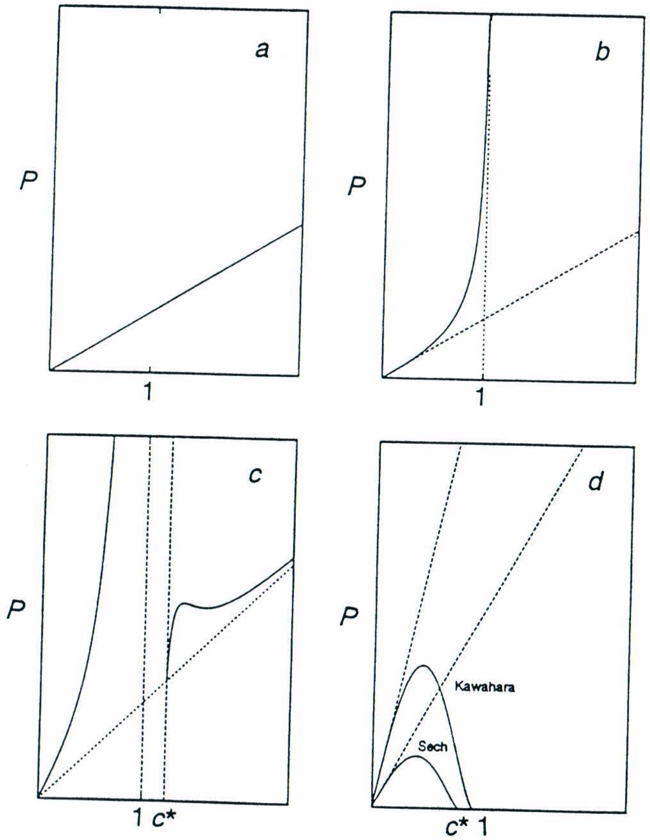

Fig. 1. Schematic of (P, $V = c$) relationship for remarkable systems: *a*, NLS (Newtonian point mechanics); *b*, SG (Lorentzian–Einsteinian point mechanics); *c*, GZ (complex point mechanics); *d*, 6GBO equation ("rocket"-like point mechanics).

2. KAWAHARA SOLITONS

This note is necessarily descriptive and we shall content ourselves, after already mentioning some cases pertaining to *microstructured solids*, with addressing the question of constructing the *point mechanics* of some solitary wavelike solutions to another system, the one that comes from the study of *phase transitions* in *shape-memory alloys* [8]. In such studies we were led to considering simultaneously higher-order nonlinearities and dispersion, as also coupling between various elastic modes. For all practical purposes, however, the following model with quadratic nonlinearity but sixth-order spatial derivatives – the *sixth-order Boussinesq* (6GBO; G for "generalized") equation – is quite sufficient [9, 10]. With an obvious notation and $u(X,t)$ meaning an elastic *strain* in spite of the notation, we have

$$u_{tt} = \left(u + u^2 + \frac{\beta}{3}u_{XX} + \frac{2\beta^2}{15}u_{XXXX}\right)_{XX} , \qquad (2.1)$$

where β may be negative or positive. Depending on the sign and value of β and the sign of the velocity parameter $\lambda = 1 - V^2$, we may obtain different *exact one-wave* solutions [10] of altogether differing shapes, e.g., (i) $\beta = -1$, $\lambda > 0$, (subsonic waves): *monotonous shapes* in sech^4; (ii) $\beta = 1$ and $0 < V < c_0 \equiv 0.866$: *oscillatory shapes* (so-called *Kawahara solitons*, see below), and (iii) $\beta = \pm 1$, $\lambda < 0$ (supersonic waves): *weakly nonlocal solitons* (so-called *nanopterons*). We refer to [10] for the analytic expressions, individual graphs, long-time evolution, and collisional behaviour of these solutions.

The pertinent questions are: (1) Are these (mathematically) true "solitons"? and (2) What is the point mechanics of the associated quasi-particles? Equation (2.1) is not exactly integrable and the collisions of the above-mentioned wavelike objects are not elastic. The first question therefore receives a definite negative answer. For the second one, we do not have the analytical means of finding these point mechanics. But (2.1), being obviously derivable from a Lagrangian (this is *nonlinear elasticity with higher-order strain gradients*), identifying the *total mass* as being a sensible measure of the wave solutions, e.g.,

$$M_0 = \int_{-\infty}^{+\infty} u(x,t)\,dx = [\bar{u}]_{-\infty}^{+\infty} , \qquad (2.2)$$

where $\bar{u}$, such that $u = \bar{u}_X$, physically is the *elastic displacement*, we can use the field-theoretic definition of wave momentum and energy and, for the above-recalled wave solutions, evaluate *numerically* the total *wave-momentum* and *energy* of these *specific* solutions. Then a best-fitting curve method is used to establish the critical relationships $P(M_0, V)$ and $E(P, M_0)$. These are presently of the *pseudo-Lorentzian* type (i.e., with a relativistic factor) but with an unusual power and a vanishing of all three quantities (mass in motion M, momentum P, and energy E) for a critical speed. Figure 2 gives these for the *Kawahara* type of solutions [case (ii) above] for which the identified exotic point mechanics is given by the following:

$$M_0 = 2.986, \quad M = M_0\gamma^{7/4},$$
$$P = M_0 V\gamma^3, \; \gamma = \left(1 - V^2\right)^{1/2}. \tag{2.3}$$

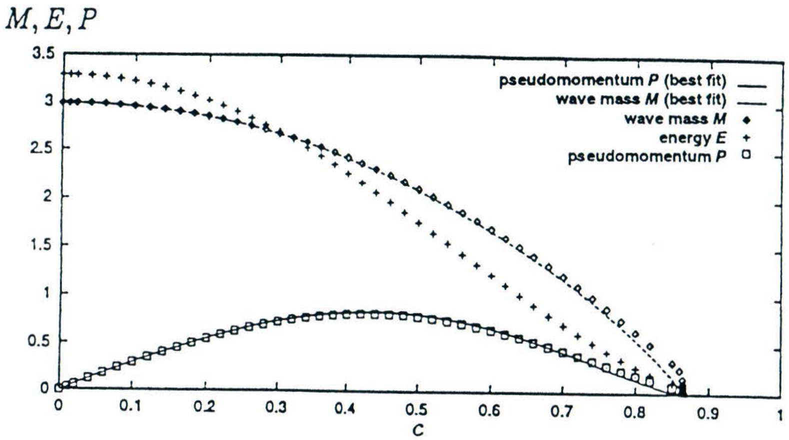

Fig. 2. Mass, momentum, and energy for Kawahara solitons for positive fourth-order dispersion $\beta = +1$ (symbols: numerical evaluation; lines: best-fit approximation).

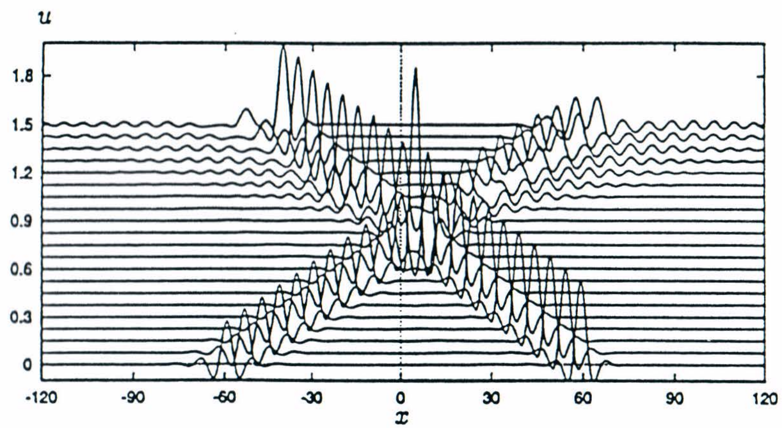

Fig. 3. Head-on collision of two Kawahara solitons with $V = 0.8$ (soliton from the left) and $V = -0.7$ (soliton from the right).

Such a mechanics reminds us of that of points with variable mass like in *rockets'* mechanics. It is very peculiar indeed and certainly not predictable by any means. Still, like all dynamics, it admits Newton's one as a limit for small velocities. A numerical scheme can be devised which conserves the basic entities of the associated point mechanics. This is used to simulate the head-on collision of two Kawahara solitons of differing velocities (Fig. 3). Here the faster solitary wave re-emerges from the inelastic collision considerably changed in shape, resembling rather a bound state of two waves. The long-time simulation, however, shows that this "bound state" finally does not survive the collision, evolving ultimately into a pulse, while the slower, but bigger wave, does preserve its identity after the collision. More practical details can be found in a lengthy paper [10] and the synthesis contribution [11].

3. CONCLUSIONS

The point of view expressed above is extremely *reductive* in that, establishing a duality of the wave process which may be multicomponent (e.g., case of SGdA and GZ systems), it still dresses the associated quasi-particle only with the essential attributes of a point particle: mass, momentum, and energy. This is all the more reductive in one space dimension where both momentum and energy are reduced to scalars that must consistently satisfy two relationships. The situation is better in more than one space dimension where the wave momentum acquires a true *co-vectorial* nature [10] and thus has a greater operational value in computing interactions of quasi-particles (e.g., changes in direction). The question is naturally raised of the existence of an *internal structure* granted to quasi-particles which would then become true complexes in themselves. Additional canonical balance laws should describe the evolution of this internal structure (spin, deformation, etc.). This is only conjectural. The balance of moment of pseudomomentum [12] is a candidate for this.

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ELASTSETE LAINETE JA KVAASIOSAKESTE MITTELINEAARNE DUAALSUS MIKROSTRUKTUURIGA MATERJALIDES

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On uuritud kvaasiosakeste käitumist diskreetses süsteemis, mille matemaatiline mudel pole täpselt integreeritav. On näidatud, et lainelevi mehhanism allub niisuguses süsteemis samadele seaduspärasustele kui elastsetes kehades. Esitatud näide käsitleb mikrostruktuuriga kristalle, kus lained on kirjeldatavad üldistatud Boussinesqi võrrandi abil. Sel juhul on üks iseloomulikke lahendeid nn. Kawahara soliton.