4

Nonlinear Duality Between Elastic Waves and Quasi-particles

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ABSTRACT Some systems governed by a set of partial differential equations present the necessary ingredients (nonlinearity and dispersion) in appropriate doses so as to become the arena of the propagation and interactions of solitary waves. In general such systems are not *exactly integrable* in the sense of soliton theory. But some of their nearly solitonic solutions can nonetheless be apprehended as quasi-particles in a certain dynamics that depends on the original system. The present chapter considers this reductive representation of nonlinear dynamical solutions for physical systems issued from solid mechanics, and more particularly elasticity with a microstructure of various origin. A whole collection of "point-mechanics" emerges thus, among which the simpler ones are Newton's and Lorentz–Einstein's. This quasi-particle representation is intimately related to the existence of *conservation laws* for the system under study and the recent recognition of the essential role played by fully *material balance laws* in the continuum mechanics of inhomogeneous and defective elastic bodies.

4.1 Introduction

As recalled in Section 4.2 the notion of *conservation laws* is traditionally associated with *hyperbolic systems* and the basic equations of *continuum mechanics* since, after WWII, it has become common practice to introduce the student to continuum physics via the statement of *global balance laws* [1] and to relate *hyperbolicity* to the existence of *local strict conservation laws.* This was due to the fact that the core of continuum physics was provided by fluid mechanics and the study of shock waves in gasdynamics, and classical elasticity (no dispersion). But with (i) the development of more complex physical and mathematical modeling — especially with the introduction of the notions of *microstructure* and *gradient theories* and the normal con-

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frontation with models issued from a sub-continuum viewpoint (e.g., lattice dynamics) — and (ii) the simultaneous discovery of nonlinear wave phenomena that cannot be so simply classified as "hyperbolic" or not, due to the existence of *dispersion* [2], the notion of *conservation laws* has acquired another range of application that is developed in the body of this chapter. The latter is mostly concerned with physical systems issued from *deformable-solid mechanics* which exhibit *both* nonlinearity and *dispersion* and, therefore, are prone to be the favorite arena of some strange *nonlinear wave phenomena* such as *solitons*.

In the last two decades, solitons have become a true paradigm of nonlinear physics and the theory of solitons has become a field of expertise in its own right [3, 4]. This will not be expanded here. But the present chapter develops the viewpoint that, even though the systems considered may not be *exactly integrable* in the sense of soliton theory, it is possible to gain insight in the dynamical behavior of many *localized nonlinear waves* by granting to them the essential attributes of point-wise particles, in a *mechanics* that depends on the original system of partial differential equations (PDEs) that governs the underlying continuum phenomenon. Thus we go *from the "local" continuum* to *the "global" discrete* while lattice dynamics goes from the *local* discrete to the *local* continuum in its long-wave approximation and numerical simulations usually go from the *local continuum* to the *local discrete*. A *localized nonlinear wave* is thus viewed as a *quasi-particle* endowed with the normal attributes of such objects, i.e., mass, momentum and energy, a point of view that clearly coincides with some recent developments in *elementary-particle physics* [5] and some causal re-interpretations of *quantum mechanics* [6]. But our point of view is *pragmatic*: we are interested in *macroscopic* phenomena and *phenomenological* physics, the conservation laws introduced in addition to classical ones being essentially used either to treat *transient* or *perturbed* evolutions of the localized nonlinear waves in question or as a *numerical means* of checking performances of numerical schemes or conservative properties of the systems when the latter are not obviously so. This type of approach which, in its style, borrows elements from *mathematical physics*, seems to be powerful and was already, but only superficially, sketched in several contributions [7]–[13]. Here it is illustrated by many examples essentially introduced by the authors and co-workers.

Section 4.2 recalls the appearance and role of conservation laws in strictly hyperbolic systems. Elements of nonlinear elasticity theory with its generalization to second-gradient of strains are given in Section 4.3 as an illustration of a good field-theoretic construct in which the *canonical balances* of energy and momentum are essential. Solitonic systems are introduced in Section 4.4 by means of examples pertaining to *exactly integrable systems*. The notion of *nearly integrable system* is then introduced in Section 4.5 together with that of perturbed global *conservation laws*. A wealth of examples issued directly from solid mechanics or re-interpreted within the framework of elasticity are given in Section 4.6. Section 4.7 sets up conclusions and the general framework concerning the *nonlinear duality* thus established between *localized nonlinear waves* and *quasi-particles.*

4.2 Hyperbolicity and Conservation Laws

From the times of Riemann, Kowaleska and Hadamard to the 1960s the theory of *conservation laws* and *hyperbolicity*, i.e., the faculty for a partial differential equation (or a system of such equations) to exhibit *dynamical* solutions traveling at a *finite speed* of propagation, have been equated. This was strengthened by the introduction of Maxwell's electromagnetic equations and the discovery of *relativity* and its inherent *causality*, as also by developments in applied mathematics, especially under D. Hilbert's influence in Germany and then R. Courant's influence in New York City. This is exemplified in *fluid dynamics* [14, 15], *magneto-hydrodynamics* [16], *elasticity* and *plasticity* [17], *magnetoelasticity* [18, 19, 20], and *electroelasticity* [21, 22]. The notion of hyperbolicity of *quasi-linear systems* and the associated *conservation laws* is masterfully dealt with in Lax [23]. For our purpose, however, we simply need simple notions that are illustrated thus.

Example I: d'Alembert (dA) equation

This is the ubiquitous one-dimensional (in space) *linear wave equation*, the *paragon* of hyperbolic systems,

$$u_{tt} - u_{XX} = 0\,, \tag{4.2.1}$$

where subscripts t and X indicate differentiation with respect to time and space respectively, and the characteristic speed which depends solely on the constitution of matter (say, of the elastic string) is here normalized to one. We all know *d'Alembert's* celebrated solution of (4.2.1) as

$$u = f(\xi) + g(\zeta)\,, \tag{4.2.2}$$

with *characteristic coordinates* $\xi = X + t$ and $\zeta = X - t$, since (4.2.1) can also be written as

$$u_{\xi\zeta} = 0\,. \tag{4.2.3}$$

Equation (4.2.1) trivially is a *conservation law* as it can be rewritten as

$$\frac{\partial}{\partial t}v - \frac{\partial}{\partial X}T = 0\,, \quad (v := u_t\,;\; T := u_X), \tag{4.2.4}$$

in which we recognize the equation of motion of a one-degree of freedom elastic body (displacement u) with density normalized to one, and "stress" T with elasticity coefficient also normalized to one. Note that u is not necessarily *longitudinal* (i.e., along the X coordinate) contrary to common thought.

It was remarked by Hayes [24] that if one multiplies (4.2.1) or (4.2.4) by u_t or u_X, and then integrates by parts, two other *conservation laws* are obtained:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2+\frac{1}{2}u_X^2\right)+\frac{\partial}{\partial X}\left(-u_t u_X\right)=0, \tag{4.2.5}$$

and

$$\frac{\partial}{\partial t}\left(-u_t u_X\right)+\frac{\partial}{\partial X}\left(\frac{1}{2}u_t^2+\frac{1}{2}u_X^2\right)=0. \tag{4.2.6}$$

We beg the reader to pay no special attention to the remarkable *symmetry* exhibited by eqs. (4.2.5) and (4.2.6), for it is an artifact of the *one-dimensionality* of the system considered. Indeed, it is sufficient to remember that in a real *elastic system* the displacement is a *vector field* with three components so that if the corresponding equation (4.2.5) remains a *scalar* equation, the corresponding equation (4.2.6) then becomes *vectorial*, and any doubt is erased. Still, we recognize in eqn. (4.2.5) the local statement of *energy conservation*, while, for the time being, we simply note that (4.2.6) is obtained by performing on (4.2.1) an operation of the space-like type while (4.2.5) is obtained by use of the corresponding time-like operation (clearly, u_t and u_X are time and space components of a unique object). Anticipating somewhat further developments, we could rewrite eqns.(4.2.5)–(4.2.6) in the following form. In addition to (4.2.4) in which we can set $p=v$ and $e=u_X$, the *linear momentum* and *strain*, respectively, we have

$$\frac{\partial}{\partial t}\mathcal{H}-\frac{\partial}{\partial X}Q=0\,, \tag{4.2.7}$$

and

$$\frac{\partial}{\partial t}\mathcal{P}-\frac{\partial}{\partial X}b=0\,, \tag{4.2.8}$$

in which we have set

$$\begin{aligned}\mathcal{H} &= u_t p-\mathcal{L}=\frac{1}{2}u_t^2+\frac{1}{2}u_X^2\,, \quad Q=Tv\,,\\ \mathcal{P} &= -u_X p\,, \quad b=-(\mathcal{L}+u_X T), \quad \mathcal{L}=\frac{1}{2}u_t^2-\frac{1}{2}u_X^2\,.\end{aligned} \tag{4.2.9}$$

The reader familiar with *elastic systems*, or *field theory* in its simplest guise, will have recognized in $\mathcal{H},\mathcal{L}$, and $\mathcal{P}$ the *Hamiltonian density*, the *Lagrangian density*, and the *canonical momentum*, while the associated fluxes Q and b may be called the energy flux or *Poynting–Umov* "vector" and the *canonical "stress"*, respectively. We see that the simplest hyperbolic system, itself a conservation law, has associated with it at least two other conservation laws. As we know, while (4.2.1) has been exploited in finding the general form of *travelling wave solutions*, the *energy conservation* (4.2.7) is practically used for implementing conservation properties e.g., in the reflection-transmission problem or as a way of assessing the performance of a *numerical scheme* in which it is demanded that a certain *measure* of the solution, the *energy norm*, (R = real line)

$$\|u\|^2 = E = \int_{\mathsf{R}} \mathcal{H}(u_t, u_X)\mathrm{d}X, \tag{4.2.10}$$

be conserved in the best possible manner (a *criterion* of performance). Nothing is usually said about eqn.(4.2.6) or (4.2.8) whose existence is ignored by most practitioners of the Art. But we can easily imagine that, in a numerical implementation, we demand the "*best* possible *conservation*" of the total *wave momentum*

$$P = \int_{\mathsf{R}} \mathcal{P}(u_t, u_X)\mathrm{d}X \,. \tag{4.2.11}$$

We shall return to this point later on. For the time being we simply note the following two points:

(i) *Quantization of linear elastic systems*:

It is a simple matter [8] to show that for a *monochromatic wave* of frequency ω and *wave number* k propagating in the linear system (4.2.1) — necessarily corresponding to low levels of energy —, if vibration energy is quantized according to the relation $\mathcal{E} = \hbar\omega$, where $\hbar$ is Planck's reduced constant, then the above-introduced *wave-momentum* $\mathcal{P}$ is also quantized according to *de Broglie's* relation of *wave mechanics*

$$\mathcal{P} = \hbar k \,. \tag{4.2.12}$$

This obviously supports the coinage of *wave-momentum* for $\mathcal{P}$ while referring to the *duality between linear waves and quasi-particles* (here *phonons*). This duality holds because $\mathcal{P}$, just like the energy but contrary to p, is *quadratic* in the first-order space-time derivatives of the field u.

(ii) Contrary to the first remark, the existence of the "extra" conservation laws (4.2.7) and (4.2.8) is not especially related to the linear nature of the starting wave equation (4.2.1). The generalization to *nonlinear hyperbolic systems* (quasi-linear systems) is straightforward as the main ingredient is the fact that the starting equation is field-theoretically based. For instance, the following one-dimensional equation that frequently appears in solid-crystal acoustics [25]:

Example 2: Quasi-linear hyperbolic equation:

$$u_{tt} - u_{XX} - \beta u_X u_{XX} = 0\,, \tag{4.2.13}$$

where β is a (generally small) *nonlinearity parameter*, enters the above-mentioned framework. We shall come back to its associated conservation laws when dealing with more general systems. But then there is even more to it, because, if this is the case, then wave systems of another nature, so-called *dispersive systems*, will also exhibit such extra conservation laws: this is where the generalization of eqn.(2.6) acquires a true operational power, indeed in the case of *nonlinear dispersive systems* that are our subject of interest in this chapter. The main result then is that the *extra conservation laws* provide a true *nonlinear duality* between certain *localized nonlinear waves* and *quasi-particles*, i.e., objects endowed with the essential attributes of a point "particle", *mass, momentum*, and *energy.* The relationships between these three quantities then depend on the starting system of wave equations. Before attacking this point, we like to comment further on the notion of *wave equation* itself as most readers would qualify as *evolution equations* some of the equations that we shall indeed consider as *wave equations.*

In view of the typical solution (4.2.2), eqn.(4.2.1) is *two-directional* in the sense that, because it involves second-order derivatives, its general solution consists of *right-running* and *left-running* solutions. The equations

$$\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial X}\right) u = 0\,, \quad \text{or} \quad \frac{\partial}{\partial \xi} u = 0\,,\ \frac{\partial}{\partial \zeta} u = 0\,, \tag{4.2.14}$$

are also *linear wave equations*, but of the *first-order* only: they exhibit *one-directional solutions.* This is also true of the following *first-order nonlinear* PDEs:

$$\left(\frac{\partial}{\partial t} \pm u \frac{\partial}{\partial X}\right) u = 0\,. \tag{4.2.15}$$

Obviously, both eqns.(4.2.14) and (4.2.15) are *conservation laws.* The second of these is *nonlinear* but, as opposed to (4.2.13), only *one-directional.* It is often used to epitomize the *shock phenomenon* in *nonlinear hyperbolic systems* using the notion of *weak solutions.* Indeed, consider the *plus* sign in eqn.(4.2.15). This is equivalent to the local conservation law:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial X}\left(\frac{1}{2} u^2\right) = 0\,. \tag{4.2.16}$$

To obtain a *weak formulation* we multiply this by an arbitrary sufficiently smooth *test function* — this is the spirit of the principle of virtual power or of the theory of *distributions* — $w(X,t)$, e.g., of class $C^1(\mathsf{R}^2)$ and of bounded support K on R^2 with $w = 0$ on the boundary ∂K of K and $w = 0$ outside K. If $u(X,t)$ is also of class C^1 with the initial condition $u(X,0) = \phi(X)$, $X \in \mathsf{R}$, then we show that a variational formulation equivalent to (4.2.16) reads

$$\int_{\mathsf{R}} \int_{t=0}^{\infty} \left(u \frac{\partial w}{\partial t} + \frac{1}{2} u^2 \frac{\partial w}{\partial X}\right) \mathrm{d}X \mathrm{d}t + \int_{\mathsf{R}} \phi(X) w(X,0) \mathrm{d}X = 0\,. \tag{4.2.17}$$

The astute point here is that all derivatives are now applied to the test function. We can therefore consider (4.2.17) as a starting point for solutions $u(X,t)$ which are only *piecewise continuous* admitting a finite discontinuity (so-called *shock*) along the *shock wave* $\Sigma(X,t)$ in the (t,X) plane; eqn. (4.2.16) will be satisfied everywhere except at Σ.

4.3 Elasticity as a field theory

Before examining in greater detail the implications of the additional balance laws accompanying *nonlinear dispersive systems*, we need to formalize somewhat the simple arguments presented at the beginning of Section 4.2. To that purpose we introduce in a nutshell the main ingredients of nonlinear elasticity theory, which provides a paradigmatic example of a field theory in which functions and parameters of the space-time descriptions are clearly differentiated from one another (for more on elasticity, see Maugin [8]).

At any regular material point $\boldsymbol{X}$, at time t, of a finitely deformable solid, in the absence of body forces and couples, there hold Cauchy's equations of motion in the Piola–Kirchhoff form (T = transpose)

$$\left.\frac{\partial}{\partial t}\boldsymbol{p}\right|_{\boldsymbol{X}} - \mathrm{div}_R\boldsymbol{T} = \boldsymbol{0}\,, \quad \boldsymbol{T}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{T}^T\,, \tag{4.3.1}$$

where the linear (physical) momentum $\boldsymbol{p}$, per unit volume of the reference configuration $\mathcal{K}_R$, the first Piola–Kirchhoff stress "tensor" $\boldsymbol{T}$ (also per unit area in $\mathcal{K}_R$, but with force components in the actual configuration $\mathcal{K}_t$), the physical velocity $\boldsymbol{v}$, and the direct-motion gradient $\boldsymbol{F}$ are defined by

$$\boldsymbol{p} = \rho_0(\boldsymbol{X})\boldsymbol{v}(\boldsymbol{X},t), \quad \boldsymbol{T} = J_F\boldsymbol{F}^{-1}\cdot\sigma\,, \quad J_F = \mathrm{det}\boldsymbol{F} > 0, \tag{4.3.2}$$

$$\boldsymbol{v} = \left.\frac{\partial\chi}{\partial t}\right|_{\boldsymbol{X}}, \quad \boldsymbol{F} = \left.\frac{\partial\chi}{\partial\boldsymbol{X}}\right|_t = \nabla_R\chi,$$
$$\left(\frac{\partial\boldsymbol{F}}{\partial t}\right)^T = \nabla_R\boldsymbol{v}, \quad \nabla_R \equiv \frac{\partial}{\partial\boldsymbol{X}}, \quad \frac{d}{dt} = \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}}, \tag{4.3.3}$$

where σ is the symmetric Cauchy stress tensor and $\boldsymbol{x} = \chi(\boldsymbol{X},t)$ is the direct motion between $\mathcal{K}_R$ and $\mathcal{K}_t$. Equations (4.3.1) may be viewed as consequences of, or statements equivalent to, the *Cauchy formulation* at any actual point $\boldsymbol{x}$, the latter being the image of $\boldsymbol{X}$ by the direct motion

$$\rho\frac{d}{dt}\boldsymbol{v} - \mathrm{div}\sigma = \boldsymbol{0}, \quad \sigma = \sigma^T\,, \quad \rho(\boldsymbol{x},t) = J_F^{-1}\rho_0(\boldsymbol{X})\,. \tag{4.3.4}$$

As $J_F > 0$ always, we can also introduce $\boldsymbol{V}$ and $\boldsymbol{F}^{-1}$, respectively the *material* velocity and inverse-motion gradient, by

$$\boldsymbol{V} = \left.\frac{\partial \chi^{-1}}{\partial t}\right|_{\boldsymbol{X}}, \quad \boldsymbol{F}^{-1} = \left.\frac{\partial \chi^{-1}}{\partial \boldsymbol{X}}\right|_{t} = \nabla \chi^{-1}, \tag{4.3.5}$$

so that we check the following relations:

$$\boldsymbol{v} + \boldsymbol{F} \cdot \boldsymbol{V} = \boldsymbol{0}, \quad \boldsymbol{F}\boldsymbol{F}^{-1} = \boldsymbol{1}, \quad \boldsymbol{F}^{-1}\boldsymbol{F} = \boldsymbol{1}_R \tag{4.3.6}$$

where $\boldsymbol{1}$ and $\boldsymbol{1}_R$ are unit dyadics in $\mathcal{K}_t$ and $\mathcal{K}_R$, respectively. In the *nondissipative* case of the *elasticity* of possibly *materially inhomogeneous* and *isotropic* solids, we can introduce a *Lagrangian function* $\mathcal{L}$ per unit volume of $\mathcal{K}_R$ and write (tr = trace)

$$\begin{aligned} \mathcal{L}(\boldsymbol{v}, \boldsymbol{F}; \boldsymbol{X}) &= \frac{1}{2}\rho_0(\boldsymbol{X})\boldsymbol{v}^2 - W(\boldsymbol{F}; \boldsymbol{X}), \\ -\frac{dW(\boldsymbol{F}; X)}{dt} &+ tr\left(\boldsymbol{T} \cdot \frac{\partial \boldsymbol{F}}{\partial t}\right) = 0, \end{aligned} \tag{4.3.7}$$

where W is the potential (elastic) energy per unit volume of $\mathcal{K}_R$. Equation $(4.3.7)_2$ reproduces Gibbs' equation for isothermal evolutions. We have then

$$\boldsymbol{T} = \left(\frac{\partial W}{\partial \boldsymbol{F}}\right)^T ; \quad \boldsymbol{f}^{\text{inh}} := \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{X}}\right)_{\text{expl}} \tag{4.3.8}$$

where the latter, the explicit gradient of $\mathcal{L}$, is a *covariant material vector.* It is the *material inhomogeneity force* which captures both *inertial* (via ρ_0) and *elastic* (via W) material inhomogeneities since, explicitly,

$$\boldsymbol{f}^{\text{inh}} = (\nabla_R \rho_0)\frac{\boldsymbol{v}^2}{2} - \left(\frac{\partial W}{\partial \boldsymbol{X}}\right)_{\text{expl}} . \tag{4.3.9}$$

By left contracted multiplication of $(4.3.1)_1$ by $\boldsymbol{v}$ or $\boldsymbol{F}^T$ and integration by parts while accounting for (4.3.7)–(4.3.9), we obtain a *scalar* conservation law, the *energy equation,*

$$\left.\frac{\partial}{\partial t}\mathcal{H}\right|_{\boldsymbol{X}} - \nabla_R \cdot \boldsymbol{Q} = 0, \tag{4.3.10}$$

and a *co-vectorial material balance law* (not in conservative form)

$$\left.\frac{\partial}{\partial t}\mathcal{P}\right|_{\boldsymbol{X}} - \text{div}_R \boldsymbol{b} = \boldsymbol{f}^{\text{inh}} \tag{4.3.11}$$

where we have defined the material *pseudomomentum* $\mathcal{P}$ and the *Eshelby* (material) stress tensor $\boldsymbol{b}$ by

$$\mathcal{P} = -\boldsymbol{F}^T \cdot \boldsymbol{p} = \rho_0 \boldsymbol{C} \cdot \boldsymbol{V}\,, \quad \boldsymbol{b} = -(\mathcal{L}\mathbf{1}_R + \boldsymbol{F}^T\boldsymbol{T})\,, \tag{4.3.12}$$

together with the following useful measures of finite strains:

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}\,, \quad \boldsymbol{C}^{-1} = \boldsymbol{F}^{-1}\boldsymbol{F}^{-T}\,, \quad \mathsf{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1}_R)\,. \tag{4.3.13}$$

Applying *objectivity* (invariance with respect to time-dependent rotations in the actual configuration) to W, we also obtain as a particular representation of W,

$$W = \bar{W}(\mathsf{E}; \boldsymbol{X}), \tag{4.3.14}$$

from which it follows that

$$\boldsymbol{T} = \mathsf{S} \cdot \boldsymbol{F}\,, \quad \boldsymbol{b} = -(\mathcal{L}\mathbf{1}_R + \boldsymbol{C} \cdot \mathsf{S})\,, \quad \mathsf{S} = \frac{\partial \bar{W}}{\partial \mathsf{E}} = \mathsf{S}^T\,, \tag{4.3.15}$$

where S is the second (symmetric, fully contravariant, material) Piola–Kirchhoff stress tensor. We then check that (A = antisymmetrization)

$$(\boldsymbol{b}\boldsymbol{C})_A = \mathbf{0} \quad \text{or} \quad \boldsymbol{b}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{b}^T\,. \tag{4.3.16}$$

The expression of eqns.$(4.3.12)_1$, $(4.3.15)_2$ and $(4.3.16)_3$ clearly demonstrates the role of *deformed metric* played by the Green finite strain $\boldsymbol{C}$. As to $\boldsymbol{b}$, on account of the condition (4.3.16), we can state that it is *symmetric with respect to* $\boldsymbol{C}$. Clearly, the material co-vector $\mathcal{P}$ and the material tensor $\boldsymbol{b}$ generalize to the three-dimensional finite-strain case the notions introduced as scalar quantities in Section 5.2. Any degree of *physical nonlinearity in strains* can be envisaged and *geometrical nonlinearities* are also included for any material symmetry. Furthermore, the above-given presentation, succinct as it, nonetheless hints at *variational formulations* [26]. Without developing these in detail here we simply record that in *elasticity* eqns.(4.3.1) and (4.3.11) are directly deduced by (i) applying a direct-motion variation (keeping $\boldsymbol{X}$ fixed - so-called *Lagrangian variation*) accompanied by the application of *Noether's theorem* [8, 27, 28] for $\boldsymbol{X}$ (parameter) — translations, or (ii) by direct variation of the inverse-motion, keeping then $\boldsymbol{x}$ fixed (so-called *Eulerian variation*) — so that (4.3.11) follows at once while $(4.3.1)_1$ is a consequence of Noether's theorem for $\boldsymbol{x}$-invariance, or else by (iii) simultaneous Eulerian and Lagrangian variations. Then $\mathcal{P}$ and $\boldsymbol{b}$ acquires the true meaning [in procedure (i)] of *canonical momentum* and *canonical stress tensor* on the material manifold. This analytical-mechanics interpretation is supported by a true canonical *Hamiltonian* formulation of dynamical finite-strain elasticity [8]. However, the definition (4.3.12) of pseudomomentum remains valid in the presence of dissipative processes as it is purely kinetic and geometric.

Its definition is generalized (see below) when there are additional degrees of freedom. Note that the $\boldsymbol{x}$-invariance is a basic one of physics whereas the $\boldsymbol{X}$-invariance strictly holds good only for *materially homogeneous bodies.* If this is the case $\boldsymbol{f}^{\text{inh}} \equiv \mathbf{0}$, and eqn.(4.3.11) becomes a true *vectorial conservation law.* Thus for homogeneous nonlinear elastic bodies, in addition to the *mass conservation* $\partial \rho_0 / \partial t = 0$, we have in all seven *conservation laws*, the three components of the *conservation of physical momentum*, the three components of the *conservation of pseudomomentum* (which is the full pullback of the former on the material manifold), and the *conservation of energy* (resulting from the application of Noether's theorem for *time* translations). This general structure is conserved for homogeneous elastic materials of *higher grade*, thanks to the underlying field-theoretical structure of the formulation. For instance, for *second-gradient* elastic materials for which the potential energy depends not only on $\boldsymbol{F}$ but also on the second material gradient $\boldsymbol{G} = \nabla_R \nabla_R \chi = \nabla_R \boldsymbol{F}$, i.e., $W = W(\boldsymbol{F}, \boldsymbol{G})$, — with $\boldsymbol{p}$ and $\mathcal{P}$ unchanged, eqns. (4.3.1) and (4.3.11) keep the same form but with $\boldsymbol{T}$ and $\boldsymbol{b}$ now given by [26]

$$\boldsymbol{T} = \bar{\boldsymbol{T}} - \mathrm{div}_R \mathsf{M}\,, \quad \bar{\boldsymbol{T}} = \left(\frac{\partial W}{\partial \boldsymbol{F}}\right)^T, \quad \mathsf{M} = \left(\frac{\partial W}{\partial \boldsymbol{G}}\right)^T, \tag{4.3.17}$$

$$\boldsymbol{b} = -(\mathcal{L}\mathbf{1}_R + \boldsymbol{F}^T \bar{\boldsymbol{T}} + 2\boldsymbol{G}^T{:}\mathsf{M}) - \mathrm{div}_{\mathbf{R}}(\boldsymbol{F}^{\mathbf{T}}\mathsf{M})\,.) \tag{4.3.18}$$

Such a modeling potentially contains both *nonlinearity* and *dispersion* effects, the latter through the obvious appearance of a length scale related to the ratio of first-order to second-order gradient effects. This modeling does not involve additional degrees of freedom. The now classical one-dimensional *Boussinesq* (BO) equation of *crystal physics* [10] belongs to this modeling.

Example 3: Elastic-crystal Boussinesq (BO) equation

$$u_{tt} - u_{XX} - 2\varepsilon u_X u_{XX} - \varepsilon\delta^2 u_{XXXX} = 0\,, \tag{4.3.19}$$

where ε is an infinitesimal parameter relating to both *nonlinearity* and *dispersion* (this is the result of a *long-wave approximation* of lattice equations) and δ is a characteristic length relating to *dispersion.* In this case, with

$$T = \bar{T} - M_X\,, \quad \bar{T} = \frac{\partial W}{\partial u_X}\,, \quad M = \frac{\partial W}{\partial u_{XX}}\,, \tag{4.3.20}$$

and

$$W = \frac{1}{2}u_X^2 + \frac{\varepsilon}{6}u_X^3 + \frac{\varepsilon\delta^2}{2}(u_{XX})^2\,, \tag{4.3.21}$$

we readily check that the (here) scalar form of the *conservation of canonical momentum* reads as in eqn. (4.2.8) but with $\mathcal{P}$ and b defined by Maugin [7]

$$\mathcal{P} = -u_X u_t\,, \quad b = -(\mathcal{L} + u_X \bar{T} + u_{XX} M) - (u_X M)_X\,. \tag{4.3.22}$$

The reduction from the general formula $(4.3.12)_1$ which correctly accounts for the operation of *pullback*, and the simple form in $(4.3.22)_1$ results from the general relationship between $\boldsymbol{F}$ and the elastic displacement $\boldsymbol{u}$:

$$\boldsymbol{F} = \mathbf{1} \cdot \mathbf{1}_S + (\nabla_R \boldsymbol{u})^T \, , \tag{4.3.23}$$

where $\mathbf{1}_S$ is a so-called *shifter* (pure change of index notation). From this and $(4.3.12)_1$ it follows that $\mathcal{P}$ and $\boldsymbol{p}$ are related by

$$\mathcal{P} + \boldsymbol{p} \cdot \mathbf{1}_S = -\rho_0 (\nabla_R \boldsymbol{u}) \cdot \frac{\partial \boldsymbol{u}(\boldsymbol{X}, t)}{\partial t} \, . \tag{4.3.24}$$

If both $\boldsymbol{p}$ and $\mathcal{P}$ are conserved, then this is also the case of the *field momentum*

$$\mathcal{P}^{\mathrm{f}} = -\rho_0 (\nabla_R \boldsymbol{u}) \cdot \frac{\partial \boldsymbol{u}}{\partial t} \, . \tag{4.3.25}$$

This indeed yields $(4.3.22)_1$ or $(4.2.9)_3$ — the *wave momentum* (a notion used by Brenig [29]) —, and we shall often call $\mathcal{P}^{\mathrm{f}}$ itself the pseudomomentum.

The above given formulation generalizes directly to the case where additional degrees of freedom are present, the space-time dependence being still indicated by the parameters $\boldsymbol{X}$ and t. As a matter of fact, with $\phi^\alpha(\boldsymbol{X}, t)$, $\alpha = 1, 2, \ldots, n$, a series of fields a priori governed by a Lagrangian density per unit volume of $\mathcal{K}_R$ in the form

$$\mathcal{L} = \bar{\mathcal{L}}(\phi^\alpha, \phi^\alpha_t, \phi^\alpha_X, \phi^\alpha_{XX}, \ldots; \boldsymbol{X}, t) \, ,$$

or limiting ourselves to first-order derivatives of the field ϕ^α and introducing a four-dimensional space-time notation

$$\mathcal{L} = \bar{\mathcal{L}}(\phi^\alpha, \partial_\mu \phi^\alpha; X^\mu) \, ,$$

where $X^\mu, \mu = 1, 2, 3, 4 = (\boldsymbol{X}, t)$ and $\partial_\mu = \partial / \partial X^\mu$, we can envisage ε-parameterized families of transformations of *both* coordinates *and* fields such that

$$(X^\mu, \phi^\alpha) \to (\bar{X}^\mu, \bar{\phi}^\alpha) \, , \quad \bar{X}^\mu = \kappa^\mu (X^\beta, \varepsilon) \, , \quad \bar{\phi}^\alpha(\bar{X}^\mu) = \Phi^\alpha(\phi^\beta(X^\nu), \bar{X}^\mu, \varepsilon) \, . \tag{4.3.26}$$

For $\varepsilon = 0$, $\kappa^\mu(X^\beta, 0) = X^\mu$, $\Phi^\alpha(\phi^\beta, \bar{X}^\mu, 0) = \phi^\alpha$. The Lagrangian $\mathcal{L}$ is assumed to transform as a *scalar density* under (4.3.26). Then we have the following results of field theory [8, 27]: (i) the fields ϕ^α are governed by the *Euler–Lagrange* variational equations

$$\frac{\delta \mathcal{L}}{\delta \phi^\alpha} = 0 \, , \quad \frac{\delta \mathcal{L}}{\delta \phi^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^\alpha)} \, , \quad \alpha = 1, 2, \ldots, n \tag{4.3.27}$$

at all points $\boldsymbol{X}$ in the material volume V, with homogeneous boundary conditions at ∂V, and (ii) for each transformation (4.3.26) which leaves the Hamiltonian action

$$A(\phi, X) = \int_{V\times\mathsf{R}} \mathcal{L}\mathrm{d}X\,, \tag{4.3.28}$$

unchanged, there is *conservation* of the *4-current* (this is the contents of Noether's theorem, i.e., $\partial_\mu \mathcal{T}^\mu = 0$),

$$\mathcal{T}^\mu = \mathcal{L}\frac{\partial \bar{X}^\mu}{\partial \varepsilon} + \sum_{\alpha=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\alpha)}\left[\frac{\partial \bar{\phi}^\alpha}{\partial \varepsilon} - (\partial_\nu \phi^\alpha)\frac{\partial \bar{X}^\nu}{\partial \varepsilon}\right]\,. \tag{4.3.29}$$

For instance, if eqns.(4.3.26) reduce to translations along X^λ, λ fixed, we can write

$$X^\mu \to \bar{X}^\mu(X^\beta \varepsilon) = X^\mu + \varepsilon \delta^\mu_\lambda\,, \tag{4.3.30}$$

where δ^μ_λ is Kronecker's symbol in four dimensions. These considerations apply directly to the cases of pure nonlinear elasticity and electromagnetoelasticity for *anisotropic* and *inhomogeneous* solids. Furthermore, by an appropriate change in notation we can use either a *direct* or an *inverse* motion description. For a *direct-motion description*, $\mathcal{L}$ is given by eqn. (4.3.7) for ϕ^α is none other than the motion mapping $\chi(\boldsymbol{X}, t)$, $\mathcal{L}$ cannot depend explicitly on $\boldsymbol{x}$ (homogeneity of physical space), and for a *materially homogeneous body*, applying the invariance (4.3.30) for *time* (t) and *space* ($\boldsymbol{X}$) translations, we obtain the *energy* and *canonical-momentum conservations* in the form

$$\partial_\mu T^\mu_{.4} = 0\,, \quad \partial_\mu T^\mu_{.K} = 0\,, \quad K = 1, 2, 3\,, \tag{4.3.31}$$

where $T^\mu_{.\nu}$ is the canonical *energy-momentum tensor* (the space-time components of the current $\mathcal{T}^\mu$). Equations (4.3.10) and (4.3.11) — with $\boldsymbol{f}^{\mathrm{inh}} = \boldsymbol{0}$ — are none other than eqns.(4.3.31)$_{1-2}$ with $\boldsymbol{b}$ the spatial part of $T^\mu_{.\nu}$, while eqn.(4.3.27)$_1$ renders the classical equation of motion (4.3.1)$_1$. There is no problem to generalize the reasoning to *higher-gradient* theories and show that the formulation (4.3.17)–(4.3.18) follows for the *nonlinear second-gradient* case. What is essential here is to notice the *additive* character of the canonical definition (4.3.29) — summation over α —, which indicates that, contrary to Euler–Lagrange equations, the canonical conservation laws pertain to the whole physical system under consideration, e.g., all degrees of freedom involved in the theory. In particular, with the above formalism, the *canonical momentum* (our *pseudomomentum*) has a canonical definition

$$\mathcal{P} = -\sum_{\alpha=1}^{n} \rho_0(\nabla_R \phi^\alpha) \cdot \frac{\partial \phi^\alpha}{\partial t}\,, \tag{4.3.32}$$

where the partial time derivative has to be replaced by a functional derivative when, although seldomly, $\mathcal{L}$ depends on higher-order time derivatives.

Equations $(4.3.12)_1$ and (4.3.23) have the typical form (4.3.32). More interestingly, this canonical form is to be found in more involved theories where the material bodies considered present a true *microstructure* governed by additional degrees of freedom (e.g., in *liquid crystals* [30]; or in elastic ferromagnets, see below). The outlined general structure did not escape the attention of some researchers in continuum mechanics [31, 32, 33] with some applications to statics, dynamics, and numerical implementation [34]–[38], but it is only recently, mainly through the efforts of the authors and co-workers [7]–[10, 13, 39]–[42], that this has been exploited in *nonlinear wave propagation* for *dispersive* systems in both analytical and numerical implementations.

4.4 Solitonic systems

Equation (4.3.19) belongs to a class of systems that we shall refer to as *solitonic systems.* Indeed, thanks to a perfect compensation between nonlinear effects [tendency to form *shocks* in quasi-linear hyperbolic systems such as (4.2.13) because points of the wave profile corresponding to different amplitudes travel at different speed] and *dispersion* [tendency to spread the waveform; the Fourier components of a signal travel at different speed], such systems admit the existence of *solitary* waves, which, in addition to being exact traveling wave solutions of narrow support (localized waves), do behave in a specific way during interactions: the past interaction of two such waves is imprinted only by a change *in phase*, no *spurious radiations* occurring during *interactions.* By analogy with the encounter of certain particles, it is then said that the interaction — or collision — is *elastic.* At this point a short historical digression may be necessary. It was J. Scott Russell [43] who first reported on a *solitary wave* ("the permanent" or "great wave") in a fluid system and reproduced this in systematic experimental investigations. This was theoretically explained by Boussinesq [44] and others (Lord Rayleigh), with a further mathematical proof of the phenomenon by Korteweg and de Vries [45] a hundred years ago while studying the *one-directional version* of the Boussinesq equation, the nowadays celebrated *KdV equation.* Passing from the BO equation to the KdV equation is effected by the method of *reductive perturbations* according to which one goes to a moving frame (say, together with the right-running coordinate of the hyperbolic part of the equation) and reasonably scales the resulting equation so that for the velocity field $v = u_t$ one has the KdV equation, a *nonlinear evolution equation* (NEE; cf. Newell [3]; Maugin et al. [22], Chapter One) in the following conventional form:

Example 4: KdV equation

$$v_t + 6vv_X + v_{XXX} = 0\,, \tag{4.4.1}$$

clearly itself a *conservation law.* In appropriate dimensionless variables, eqn. (4.4.1) admits a *solitary wave* solution of the universal form

$$v = \operatorname{sech}^2(X - t)\,, \tag{4.4.2}$$

while (4.3.19) admits *supersonic* (with respect to the characteristic speed *one*) *solitary wave* solutions of the *tanh* type. The solitonic nature, or *particle-like* behavior, of the localized wave solutions of the *KdV system* was shown by Zabusky and Kruskal [46], an occasion on which they coined the term *solitons* in accord with common usage in naming particles in physics — see Kruskal [47]. It is only with that discovery that the individualized (permanent) wave captured for good the attention of many investigators and solitons became an important field of modern nonlinear physics. From there on the KdV equation served as the featuring example, together with some other canonical systems (see below), of the *integrability theory* and of techniques for generating *multiple-soliton* solutions. Thus the *method of inverse scattering* (IS) was developed for that purpose by Kruskal and co-workers and Zakharov and co-workers (e.g., Zakharov and Shabat [48]). Simultaneously, it was found that the mathematical property of *exact integrability,* equivalent to the solution–construction scheme by IS, was accompanied by the existence of an *infinite hierarchy of conservation laws* for exact solitonic systems. Such relatively simple systems are listed in Calogero and Degasperis [49].

While the first few conservation laws associated with the KdV equation were found by trials and errors, there now exist algorithms to construct systematically these conservation laws for the most well-known equations [50]. According to Noether's theorem, these conservation laws are related to generalized symmetries of the Lagrangian or Hamiltonian density from which the integrable PDEs derive [51]. But we are not so much interested in that because nobody has ever *really* exploited these conservation laws beyond the second or third one. An essential reason for this is that if we, in effect, want to grant to a localized nonlinear wave the *attributes* of a *point* particle, then this particle — apart from electric charge and quantum numbers-can only have *three attributes* in the one-dimensional case (five in the three-dimensional case), which are *mass* M_0, *momentum* P (three components in the three-dimensional case: $\boldsymbol{P}$), and energy E, and the latter two are necessarily related in some way in the *point-mechanics* of interest: the *Emperor* is almost *naked.* Still, as we shall see, this is enough to generate interesting *methods of perturbation* of exactly integrable systems and *criteria of qualification* for numerical schemes.

The *conservation laws* of interest are none other than those of *energy* and *wave momentum* (pseudomomentum or canonical momentum). As we are especially interested in systems issued from *elasticity theory,* or systems which can be re-interpreted in such a framework, then we now see the relevance of the developments in Sections 5.2 and 5.3, as we really work towards establishing a *nonlinear duality* between "nonlinear elastic dispersive systems" and their "quasi-particle associates" through the notion of localized waves. Of course, a duality between soliton-like solutions of some systems issued from quantum physics and *elementary particles* was rapidly established by nuclear and high-energy physicists [5]. We must also remember the attempts of de Broglie and Bohm to reconcile quantum physics and a causal interpretation by introducing, in a nonlinear framework, the notion of *pilot wave* guiding the amplitude of the probability ($|\psi|^2$) of presence of a particle as a wave of singularity for which *conservation laws* and a *hydrodynamic analogy* play an essential role (see pp. 113–124 in Holland [6] — also Jammer [52]). It is possible that the present developments bear some relationship to this, but we emphasize that we are mostly interested in *macroscopic problems* issued from *engineering sciences* and *phenomenological physics*. However, to illustrate our endeavour, we shall first consider some simple paradigmatic exactly integrable systems that appear in many instances.

The first remark concerns the *integrable system* (4.4.1) which itself is a conservation law but, as remarked by Maugin [39], admits next-order conservation laws that are not entirely unambiguously constructed. Indeed, while (4.4.1) also reads

$$\frac{\partial}{\partial t}v + \frac{\partial}{\partial X}(3v^2 + v_{XX}) = 0\,, \tag{4.4.3}$$

a straightforward application of a powerful algorithm (Ablowitz and Segur [50], p.56) yields the next-order conservation law

$$\frac{\partial}{\partial t}\left(v^2 + v_{XX}\right) + \frac{\partial}{\partial X}\left(4v^3 + 8vv_X + 5v_X^2 + v_{XXXX}\right) = 0\,. \tag{4.4.4}$$

But in early studies [53] of the KdV equation, when such algorithms did not exist, it was proposed to consider the conservation law

$$\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right) + \frac{\partial}{\partial X}\left(2v^3 + vv_{XX} - \frac{1}{2}v_X^2\right) = 0\,. \tag{4.4.5}$$

But this is indeed related to (4.4.4) if we note that the second X-derivative of (4.4.3) can be rewritten as

$$\frac{\partial}{\partial t}v_{XX} + \frac{\partial}{\partial X}\left(6v^3 + 6vv_X + v_{XXXX}\right) = 0\,, \tag{4.4.6}$$

itself a conservation law, and then a mere addition of twice (4.4.5) and once (4.4.6) renders (4.4.4). This remark is somewhat annoying because it shows the existence of an infinity of conservation laws containing the contribution v^2 in the conserved quantity. Furthermore, a *false symmetry* between successive conservation laws can be built, being merely an artifact of the one-dimensionality in space of the considered system. For instance, an *ad* *hoc* linear combination of (4.4.5) and (4.4.6) yields the conservation of the quantity $(3v^2 + v_{XX})$, which happens to be the *flux* present in the first conservation law (4.4.3). The third conservation law in the line of (4.4.3) and (4.4.5) — not (4.4.4) — obtained by multiplying (4.4.2) by $3v^2$ and rearranging terms, reads (cf. Bathnagar [54], p.126)

$$\frac{\partial}{\partial t}\left(v^3 - \frac{1}{2}v_X^2\right) + \frac{\partial}{\partial X}\left(\frac{9}{2}v^4 + 3v^2 v_{XX} + \frac{1}{2}v_{XX}^2 + v_X v_t\right) = 0\,. \qquad (4.4.7)$$

The logic and usefulness of the various conservation laws recalled so far for KdV systems are not obvious. The first conserved quantity from eqn.(4.3) hints at introducing the *potential* $\bar{u}$ of v by $v = \bar{u}_X$, so that we introduce a conserved *mass* M_0 by

$$M_0 = \int_{\mathsf{R}} v \mathrm{d}X = [\bar{u}]_{-\infty}^{+\infty}\,, \qquad (4.4.8)$$

where [..] denotes the difference (the "jump") between values of the enclosure at the two infinities. That is, M_0 may alternately be qualified of *difference of potential* or, in electrical terms, of *"voltage"* of the solution. This seems to be a satisfactory gross "measure" of the solution. It would then seem that v^2 would be a good local measure of the energy. Is this really the case? The answer is no. Indeed, we may consider the total *Hamiltonian*

$$H = -\int_{\mathsf{R}} (v^3 - \frac{1}{2}v_X^2)\mathrm{d}X\,. \qquad (4.4.9)$$

With (v, u) playing the role of Hamiltonian variables usually denoted (p, q), the first of Hamilton's equation $q_t = \delta H/\delta p$, where $\delta/\delta p$ denotes the functional derivative, yields the compatibility condition $\bar{u}_{Xt} = v_X$, while the second of Hamilton's canonical equation, $v_t = -\partial(\delta H/\delta v)/\partial X$, is none other than the KdV equation. Thus energy *conservation* appears to be related to the conservation law (4.4.7) rather than to (4.4.5) or (4.4.4). Now if we consider $v = \bar{u}_X$ as in (4.4.8), then the total *canonical momentum* of the v-motion should read (compare to $(4.3.22)_1$)

$$P = -\int_{\mathsf{R}} u_X u_t \mathrm{d}X\,. \qquad (4.4.10)$$

But we immediately check that $\bar{u}$ satisfies the NEE

$$\bar{u}_t + 3\bar{u}_X^2 + \bar{u}_{XXX} = 0\,. \qquad (4.4.11)$$

On account of this, P transforms to

$$P = \int_{\mathsf{R}} (3\bar{u}_X^3 + \bar{u}_X \bar{u}_{XXX})\mathrm{d}X = \int_{\mathsf{R}} (3v^2 + v v_{XX})\mathrm{d}X\,, \qquad (4.4.12)$$

and we can identify the already cited linear combination of eqns.(4.4.6) and (4.4.7) as the correct local *conservation of momentum* for localized wave solutions of the KdV equation. This reasoning is tantamount to saying that, insofar as the "quasi-particle" description of one-directional wave equations (NEE) is concerned, the basic form is to be found in the original two-directional wave equation, here the BO equation. Indeed, let us consider anew the BO equation in its "good" or "improved" guise as proposed by several authors [55, 56].

Example 5: "Good" Boussinesq (GoB) equation

$$u_{tt} - u_{XX} - (u^2 - u_{XX})_{XX} = 0 . \tag{4.4.13}$$

Introducing the auxiliary variables q and w, this can be rewritten as the *Hamiltonian system* [58, 59]

$$\begin{aligned} u_t &= q_X , \\ w &= u_X , \\ q_t &= w_X^2 + w_{XX} - w , \end{aligned} \tag{4.4.14}$$

in which the first two are mere definitions of q and w. The mass M, momentum P, and energy E of "soliton" solutions of (4.4.13) or (4.4.14) are given by

$$M = \int_{\mathsf{R}} u \mathrm{d}X , \tag{4.4.15}$$

$$P = -\int_{\mathsf{R}} uq \mathrm{d}X , \tag{4.4.16}$$

$$E = \frac{1}{2}\int_{\mathsf{R}} \left(q^2 + w^2 + u^2 + \frac{2}{3}u^3\right) \mathrm{d}X . \tag{4.4.17}$$

As the system considered is exactly integrable, the quantities just defined are strictly conserved. But their expressions may look somewhat awkward. However, introducing the potential $\bar{u}$ by $u = \bar{u}_X$, with the condition $\bar{u}(X \to -\infty) = 0$, it is verified that

$$M = [\bar{u}]_{-\infty}^{+\infty} , \quad \bar{u}_t = q , \quad uq = \bar{u}_X \bar{u}_t , \quad \frac{1}{2}q^2 = \frac{1}{2}\bar{u}_t^2 , \tag{4.4.18}$$

so that M has the same interpretation as in the KdV case, while P and E indeed take their canonical definitions in terms of the potential $\bar{u}$. Simultaneously, in terms of elasticity theory, it is $\bar{u}$ that has the meaning of a *displacement* while u is a strain *per se*. But, accepting the philosophy presented in Sections 5.2 and 5.3 one can also consider eqn.(4.4.13) as a field equation issued from second-grade nonlinear elasticity, and multiply it by

u_X and integrate by parts to arrive at the *pseudomomentum conservation law* [7]

$$\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial b}{\partial X} = 0\,, \quad \mathcal{P} \equiv -u_X u_t\,, \tag{4.4.19}$$

and b given as in (4.3.22) up to the notation and some signs. By space integration over R and for vanishing field derivatives at infinity, we obtain the global balance law

$$\frac{dP}{dt} = 0\,, \quad P = \int_{\mathsf{R}} \mathcal{P} dX\,. \tag{4.4.20}$$

Example 6: sine-Gordon (SG) equation
This is the one-dimensional (in space) PDE

$$u_{tt} - u_{XX} - \sin u = 0\,, \tag{4.4.21}$$

where both nonlinearity and dispersion are contained in the sin function. This ubiquitous equation which can also be rewritten as *Enneper's equation* of surface geometry (compare to (4.2.3))

$$u_{\xi\zeta} - \sin u = 0 \tag{4.4.22}$$

by introducing right- and left-running characteristic coordinates, appears in many fields of science, especially while studying the structure of magnetic domain walls [20] and Josephson junctions [59]. From the mechanical viewpoint, such an equation can be obtained while studying the *torsion* of some elastic bars [60] and, above all, as an elementary model of *dislocation motion* [61]. It is *exactly integrable* and, remarkably, *Lorentz invariant.* It admits *subsonic* solitary-wave solutions of the *kink* form

$$u(X,t) = \bar{u}(\xi) = 4\tan^{-1}(\exp \pm\gamma(\xi - \xi_0)) \tag{4.4.23}$$

wherein

$$\xi = X - ct\,, \quad \gamma = (1-c^2)^{-1/2}\,, \quad |c| < 1\,. \tag{4.4.24}$$

Viewed as an *elastic system* eqn.(4.4.21) is derivable from the following *Lagrangian-Hamiltonian* framework where the sinusoidal term should be interpreted as the effect of an *external source* (periodic substrate for a one-dimensional elastic atomic chain) since the classical elastic energy cannot depend explicitly on u:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}u_t^2 - \frac{1}{2}u_X^2 - (1-\cos u)\,, \quad p = \frac{\partial \mathcal{L}}{\partial u_t}\,, \\ \mathcal{H} &= pu_t - \mathcal{L} = \frac{1}{2}(p^2 + u_X^2) + 2\sin^2(u/2)\,, \end{aligned} \tag{4.4.25}$$

with Hamiltonian equations

$$u_t = \frac{\partial \mathcal{H}}{\partial p} = p\,, \quad p_t = -\frac{\delta \mathcal{H}}{\delta u} = u_{XX} - \sin u\,. \tag{4.4.26}$$

A *kink* (2π solution in u) or an *anti-kink* (-2π solution) may be considered as a *quasi-particle* with *rest mass* M_0, *momentum* P and *energy* E given by

$$M_0 = 8 = E(0), \tag{4.4.27}$$

$$P = \int_{\mathsf{R}} (-u_X u_t) \mathrm{d}X = 8\gamma c = Mc = \frac{M_0 c}{(1-c^2)^{1/2}}\,, \tag{4.4.28}$$

$$E = \int_{\mathsf{R}} \mathcal{H} \mathrm{d}X = 8\gamma = E(c)\,, \tag{4.4.29}$$

with the classical relativistic relationship between M_0, P and E:

$$E^2(c) = M_0^2 + P^2(c)\,. \tag{4.4.30}$$

One can also define a "*charge*"

$$q = (2\pi)^{-1} \int_{\mathsf{R}} u_X \mathrm{d}X = \frac{1}{2\pi}[u]_{-\infty}^{+\infty} = \pm 1\,, \tag{4.4.31}$$

the sign depending on the sense of "rotation" of u. Thus *kinks* and *antikinks* may be viewed as *relativistic point- (quasi)-particles.* Note that the solutions (4.4.23) exist in *statics* and that apart from the limitation in speed given by the last of (4.4.24), the "amplitude" of the localized wave is not related to the speed c; such solutions are called *topological solitons.* However, q provides an algebraic "measure" (the "helicity" or *screw-sign*) of the solution. No wonder that kinks and antikinks have attracted so much attention with *particle physicists.*

Example 7: Nonlinear Schrödinger (NLS) equation

This is an equation which typically governs the complex small amplitude of localized modulated signals in many nonlinear systems, in particular *nonlinear optics.* The quantum origin of the denomination is obvious. For a cubic nonlinearity and appropriate scaling this *exactly integrable* equation [48] reads

$$i\psi_t + \psi_{XX} + 2\lambda|\psi|^2\psi = 0\,, \tag{4.4.32}$$

where $\psi(X,t)$ is the complex amplitude and λ is a real scalar parameter. The *mass, momentum,* and *energy* of *bright soliton solutions* (see Drazin and Johnson [4] for this notion) are given by [62]

$$M = \int_{\mathsf{R}} |\psi|^2 \mathrm{d}X\,, \tag{4.4.33}$$

$$P = \int_{\mathsf{R}} i(\psi\psi_t^* - \psi^*\psi_t)\mathrm{d}X\,, \tag{4.4.34}$$

$$E = \int_{\mathsf{R}} \frac{1}{2}(|\psi_X|^2 - \lambda|\psi|^4)\mathrm{d}X\,, \tag{4.4.35}$$

where * indicates *complex conjugacy.* The quantum physicist will recognize in (4.4.33) the total probability of presence of a particle of *wave function* ψ according to Max Born's interpretation — this should be normalized to *one.* The *canonical momentum* P was introduced in the causal re-interpretation of quantum mechanics by Takabayashi (cf. Holland [6], p.113) by treating ψ as a classical complex field. In other circumstances, M may be interpreted as the *total wave action* or the *total number of phonons* [63, 64]. All above introduced systems govern only one-*degree of freedom.* As an example of an *exactly integrable two-degrees of freedom* systems, which are very few indeed, we have the following.

Example 8: Zakharov (Z) system

This, in one dimension of space, couples a complex-valued field a and a real-valued scalar field, n as follows [65]:

$$ia_t + a_{XX} = na\,, \tag{4.4.36}$$

$$n_{tt} - c_T^2 n_{XX} = 2(|a|^2)_{XX} \tag{4.4.37}$$

where c_T is the characteristic speed of the n subsystem. The latter is linear in n, so that the dispersion comes from the a-system and the nonlinearity from the coupling. For one-soliton solutions, it is clear that the system (4.4.36) is equivalent to a cubic NLS equation. The *mass* may be defined just as in the NLS case and the momentum, recalling the additive nature of *wave momentum* for several degrees of freedom, is obtained by combining expressions of the type (4.4.34) and $(4.4.19)_2$:

$$M = \int_{\mathsf{R}} |a|^2 \mathrm{d}X\,, \quad P = \int_{\mathsf{R}} \{i(aa_t^* - a^*a_t) - n_X n_t\}\,\mathrm{d}X\,. \tag{4.4.38}$$

We shall come back to the energy E in a more complex case.

All systems considered so far are *exactly integrable* from the point of view of *soliton theory*, and only in these systems can we obtain an exhaustive analytical description of rather special cases. Only very simplified physical models can correspond to such systems. It takes only a small step in direction in making the physical modelling more realistic and the integrability (or at least the analytical form of the solutions) is lost. Such more realistic systems, although not exactly integrable, may be *close* to such a property, in the sense that *deviations from a pure solitonic behavior* may be studied by means of *perturbation techniques*, and taking benefit of the existence of *slightly perturbed conservation laws.* This is the expectation in many macroscopic systems that, following Kivshar and Malomed [63], we refer to as *nearly integrable systems.*

4.5 Nearly integrable systems

Typically, the perturbed *sine-Gordon* (PSG) equation

$$u_{tt} - u_{XX} - \sin u = \varepsilon f(u, u_t, ..; X, t) \tag{4.5.1}$$

is such a system, the *double sine-Gordon* (DSG)

$$u_{tt} - u_{XX} - \sin u - \varepsilon \sin 2u = 0\,, \tag{4.5.2}$$

being a special case of (4.5.1). More generally, all systems of the generic form [9]

$$\begin{aligned} NL(\phi) &= \sum_{\alpha}^{N} \eta_\alpha g_\alpha(\Phi, U^\beta) + \varepsilon h(\Phi), \\ W_\alpha(u^\alpha) &= \eta_\alpha f_\alpha(\Phi) + \varepsilon k(U^\beta)\,, \quad \alpha, \beta = 1, 2, \ldots, N\,, \end{aligned} \tag{4.5.3}$$

where Φ and U^β stand for $\{\phi, \phi_t, \phi_X, \ldots\}$ and $\{u^\beta, u_t^\beta, u_X^\beta, \ldots\}$, ε and η_α are small parameters, W_α are linear wave-like (d'Alembert) operators and NL represents the *nonlinear dispersive wave* operator that gives rise to an exactly integrable system for h, k and all g_α and f_α zero, belong to the class of *nearly integrable systems* for which the methods exposed hereinafter apply. We shall illustrate our aim by means of examples obtained while solving problems in *continuum mechanics*, noting that for *all systems derived from a Lagrangian–Hamiltonian* framework we have *local conservation laws* for mass, "physical" linear momentum, energy and pseudomomentum and for pure solitonic systems, and soliton solutions (with all field derivatives vanishing at infinities and by integration over space), we have *global conservation* of *"mass"*, *canonical momentum*, and *energy* in the symbolic form:

$$\frac{dM}{dt} = 0\,, \quad \frac{dP}{dt} = 0\,, \quad \frac{dE}{dt} = 0\,, \tag{4.5.4}$$

and there exists a definite relationship between E and P such that

$$E = \bar{E}(M, P)\,. \tag{4.5.5}$$

If the quasi-particle associated with the exact solitonic system is *Newtonian*, then $P = Mc$ and $E = P^2/2$. If it is *Lorentzian*, we have the relationships given in eqns.(4.4.28) and (4.4.30). This point-particle behavior, i.e., the type of *mechanics* satisfied by moving quasi-particles, depends on the starting system of PDEs. Some nearly integrable systems may have associated with them quasi-particles that exhibit relationships $P(M,c)$ and $\bar{E}(M,P)$ which belong to a new type of mechanics. We shall see examples of this somewhat extraordinary fact. As a matter of fact, we expect that for such systems, we have *perturbed* forms of eqns.(4.5.4) in the symbolic form

$$\frac{dM}{dt} = 0\,, \quad \frac{dP}{dt} = \mathcal{F}\,, \quad \frac{dE}{dt} = Q\,, \tag{4.5.6}$$

where $\mathcal{F}$ and Q are, respectively, a *global "force"* driving the soliton and a heat supply. With, e.g., $\mathcal{F}$ expressible in terms of P and the *velocity* c, the integration of $(4.5.6)_2$, whether analytical (in very few cases, obviously) or numerically, will yield $P(M,c)$, hence the new type of mechanics!

4.6 Examples from Continuum Mechanics

As a first example we consider a two-degrees-of-freedom — but still one-dimensional — system obtained while studying the dynamics of a linear elastic continuum endowed with an internal degree of freedom of rotation, either magnetic spin in elastic ferromagnets [66], or a *rotational electric dipole* in elastic ferroelectic crystals [67, 68] or in a pure mechanical model with rotational degrees of freedom (so-called *oriented media* [69]–[71]). This is the *sine-Gordon–d'Alembert* system according to the coinage given by Kivshar and Malomed [63]. For the sake of simplicity we consider that the rotational degree of freedom couples only with one translational degree of freedom (transverse displacement).

Example 9: sine-Gordon–d'Alembert (SGdA) system

Let ϕ be twice the angle of rotation (say, of magnetic spins in a plane parallel to the X-axis of propagation and the polarization of the transverse elastic displacement u; so-called *Néel wall* in ferromagnetism). In the appropriate scaling this system reads

$$\begin{aligned} \phi_{tt} - \phi_{XX} - \sin\phi &= \eta u_X \cos\phi\,, \\ u_{tt} - c_T^2 u_{XX} &= -\eta(\sin\phi)_X\,, \end{aligned} \tag{4.6.1}$$

where c_T is a characteristic elastic speed and η is a coupling parameter (magnetostriction in ferromagnetism, electrostriction in electroelasticity). The system couples a *sine-Gordon* equation and a *linear wave* equation, and thus deserves its name. This system is not *exactly integrable* from the point of view of soliton theory because the u-subsystem induces *radiations* during soliton interactions [68], but exact one-soliton solutions are known analytically [67]. For all practical purposes, ignoring the physical origin of the function ϕ, we may consider (4.6.1) as a two-degrees of freedom *nonlinear dispersive elastic* system with displacement components u and ϕ. Using the canonical formalism and noting the additive nature of some properties — cf. eqns.(4.3.29) and (4.3.32) —, we have the following canonical expressions:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(u_t^2 + \phi_t^2) - W(u_X, \phi_X, \phi)\,, \\
W &= \frac{1}{2}(\phi_X^2 + c_T^2 u_X^2) + \eta u_X \sin\phi - (1 + \cos\phi)\,, \\
\mathcal{H} &= p_u u_t + p_\phi \phi_t - \mathcal{L}\,, \quad p_u = u_t\,,\ p_\phi = \phi_t\,, \\
T &= \frac{\partial W}{\partial u_X}\,, \quad \mu = \frac{\partial W}{\partial \phi_X}\,,
\end{aligned} \tag{4.6.2}$$

and

$$\mathcal{P} = -(\phi_t \phi_X + u_t u_X)\,, \quad b = -(\mathcal{L} + u_X T + \phi_X \mu)\,, \tag{4.6.3}$$

so that there hold the local and global balance equations of wave momentum in the now *canonical form*:

$$\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial b}{\partial X} = 0\,, \quad \frac{dP}{dt}(\mathsf{R}) = [b]_{-\infty}^{+\infty}\,, \quad P := \int_{\mathsf{R}} \mathcal{P} \mathrm{d}X\,. \tag{4.6.4}$$

For solitary-wave solutions for which all field derivatives vanish at infinity, b vanishes at infinity and P is conserved. However, if, for instance, an applied magnetic field or electric field manifests itself through a torque acting on magnetic or electric dipoles, this, in turn, results in the presence of a perturbing "force" $\varepsilon f(X, \phi)$ in the right-hand side of eqn.$(4.6.1)_1$. Then, the conservation of P will be replaced by an equation of the type $(4.5.6)_2$ amenable by perturbation theory. The same remark applies to the energy equation (cf. eqn.$(4.5.6)_3$). This method of "balance equations" — see also Malomed [72] — was exploited by Pouget and Maugin [73] and Kivshar and Malomed [74], to study the *varied* motion of *magnetoacoustic domain walls* under the influence of an applied field. That is, the variations of the energy E and of the momentum P are established in terms of the external agents. By reason of the basic Lorentz invariance of the sine-Gordon equation, the result is *not very different* from the *uniformly accelerated motion* of a point particle in a constant field of force. This example has another peculiarity in the ferromagnetic case. The true *spin-precession* equation in the three-dimensional case exhibits the *gyroscopic nature* of magnetic spin: the magnetic spin does not produce any power in a real precession. Using a classical field approach, this means that no quadratic kinetic energy is associated with magnetic spin density (cf. Fomethe and Maugin [75] who study canonical three-dimensional conservation laws for elastic ferromagnets). But still the latter exhibits *wave momentum* and *kinetic energy* in eqns. (4.6.3). The simple reason for this is that in establishing eqns. (4.6.1) an adiabatic elimination of the out-of-plane component (that remains small) of the vectorial spin has been achieved so that the system becomes a classical one for the remaining angle ϕ.

Example 10: Generalized Boussinesq (GBO) systems

There are several ways to generalize the BO equation, obviously all leading to *nonexactly integrable systems.* One such system is obtained while studying the ferro- elastic-phase transition as a dynamical process in elastic crystals [76, 77]. Here we briefly examine the generalization of this modeling proposed by the authors [41] when approaching the difference scheme of lattice dynamics in a more accurate way. With $s = v_X$ a *shear strain*, from a lattice-dynamics approach and a long-wavelength limit while neglecting couplings with other strain components, one obtains the following type of equation:

$$s_{tt} - c_T^2 s_{XX} - [F(s) - \beta s_{XX} + s_{XXXX}]_{XX} = 0\,, \qquad (4.6.5)$$

where F is a polynomial in s starting with second degree (e.g., a nonconvex function admitting three minima) c_T is a characteristic speed and β is a positive scalar. We may say that both the nonlinearity *and* dispersion have been increased compared to the classical BO equation. Equation (4.6.5) is *stiff* in the sense that it involves a *sixth-order* space derivative, a situation that obviously imposes rather strong limit conditions at infinity or at the ends of a finite interval in numerical computations. Equation (4.6.5) may also be referred to as the *sixth-order generalized Boussinesq equation* (6GBO) [41, 78]. In spite of its apparent complexity eqn.(4.6.5) admits *solitary-wave* solutions [41, 79] which involve the ubiquitous *sech* function (but to the fourth power) for a single value of the phase speed — the existence of different solitary-wave solutions with a continuous spectrum for c was shown numerically [78]. But, it is true, for a velocity too close to c_T, these are not able to preserve their shape and eventually they transform into *pulses* which, in turn, exhibit a *self-similar* ("Big Bang") behaviour as the amplitude of the pulse decreases with time while its support increases (a phenomenon analogous to a *red shift*). These pulses practically pass through each other without changing qualitatively their shapes — save the red-shifting — with perfect conservation of "mass" and "energy", so that these pulses may qualitatively be claimed to be "solitons" [41]. The "mass", "momentum" and "energy" of system (4.6.5) can be defined thus. Let $F(s) = -dU(s)/ds$. First we rewrite (4.5.5) as a *Hamiltonian system* by introducing the triplet (s, q, w) such that

(4.6.5) is equivalent to

$$\begin{aligned} s_t &= q_{XX}\,, \\ w &= s_{XX}\,, \\ q_t &= c_T^2 s + F(s) - bw + w_{XX}. \end{aligned} \tag{4.6.6}$$

Then we have

$$M = \int_{\mathsf{R}} s\mathrm{d}X = [v]_{-\infty}^{+\infty}\,, \tag{4.6.7}$$

$$P = -\int_{\mathsf{R}} sq_X \mathrm{d}X = -\int_{\mathsf{R}} v_X v_t \mathrm{d}X\,, \tag{4.6.8}$$

$$\begin{aligned} E &= \int_{\mathsf{R}} \frac{1}{2}\left\{c_T^2 s^2 + q_X^2 - 2\mathcal{U}(s) + \beta s_X^2 + w^2\right\} \mathrm{d}X \\ &= \int_{\mathsf{R}} \frac{1}{2}\left\{c_T^2 s^2 + v_t^2 - 2\mathcal{U}(s) + \beta s_X^2 + w^2\right\} \mathrm{d}X \end{aligned} \tag{4.6.9}$$

where we assumed that $v_t(-\infty) = 0$, so that the transformations indicated in the second parts of (4.6.8) and (4.6.9) hold good. Then there hold the global balance laws

$$\frac{dM}{dt} = \frac{dE}{dt} = 0, \tag{4.6.10}$$

and

$$\frac{dP}{dt} = \mathcal{F} = -\left[s_{XX}^2\right]_{-\infty}^{\infty}. \tag{4.6.11}$$

For solitary-wave solutions, the "driving" pseudo-force $\mathcal{F}$ (one of those "material" and "configurational" forces heralded by Maugin [8, 80]) is in fact equal to zero by virtue of the asymptotic boundary conditions. In numerical applications the infinite interval is replaced by a large but nonetheless *finite* interval $[-L_1, +L_2]$, and we see that the lack of exact satisfaction of limit (boundary) conditions at the interval ends is equivalent to the presence of a *driving force* acting on the solitary-wave solution or on its "quasi-particle" interpretation. But the global conservation laws (4.6.10)–(4.6.11) then still apply with the appropriate boundary conditions at the ends of the interval ($s = 0, s_X = 0, q_X = 0$). A strongly implicit conservative scheme [78] is used in order to always preserve both M and E. The pseudoforce (4.6.11) is felt only when the solitons "hit" the boundaries and rebound from them. Yet the energy remains unchanged.

The following remarks are in order concerning the 6GBO equation.

The *wavicle* dynamics of 6GBE is dominated by its *pseudo*-Lorentzian (in fact *anti*-Lorentzian) character. In the "real" Lorentzian dynamics the mass and momentum of a particle increase with the increase of velocity and eventually become infinite at the characteristic speed c_0 (speed of light). Because of their sub-luminous nature, the localized waves of 6GBE have amplitudes that decrease with the increase of their phase velocity (*celerity*) c and eventually decay to zero at the characteristic speed c_0. Yet their dynamics resembles the Lorentzian one in the sense that the factor

$$\gamma = \sqrt{1 - \frac{c^2}{c_0^2}}$$

enters the picture. It is clear that contrary to the Lorentzian dynamics, it must enter the formulas with positive powers, because for $c \to c_0$ all of the quantities must decay to zero.

This *anti*-Lorentzian dynamics appears to be characteristic of all of the different equations of the Boussinesq Paradigm which contain only spatial derivatives for the dispersion and are at the same time linearly stable. However, the 6GBE offers two different families of sub-luminous localized waves. This was recently examined in a work [78] where we employed different numerical techniques for its simulation. Here we make use of the already developed algorithms and perform systematic computations so that we obtain an extensive set of data for the *mass*, *pseudomomentum* and *energy* of the solitary waves. As there appear two different families of localized waves we discuss them here separately.

The monotone sech-like shapes

These shapes appear when the fourth-order dispersion is negative, i.e., when it acts together with the sixth-order dispersion. In fact for $c > \sqrt{0.75} \approx 0.866$ they are strictly monotone because the tails decay exponentially. For $c \leq \sqrt{0.75}$ they have oscillatory damped tails but of extremely low amplitude so that the fact that the shapes are not strictly monotone cannot be discerned on the graphs with normal scales for the variables. In this case the celerity of the localized waves range from zero to the characteristic velocity c_0, the latter being equal to unity.

Since there is no analytic expression for the mass, pseudomomentum and energy we tried to find the best approximation containing constants, and powers of the celerity c and "Lorentzian" factor γ. We restricted ourselves by taking only powers of γ that are integer multiples of $\frac{1}{2}$, $\frac{1}{3}$ or $\frac{1}{4}$ (see [78] for the details). The best fit obtained under these constraints reads

$$M = M_0 \gamma^{\frac{5}{4}}, \quad M_0 = 7.4, \qquad P = Mc\gamma^2 = M_0 c \gamma^{\frac{13}{4}}. \tag{4.6.12}$$

We attempted some best fit approximations for the energy too, but due to the non-convexity of the latter the number of possible different combinations of powers of c, γ, M, and P increases to such an extent that renders impossible the task to choose one expression over another because quantitatively they fit equally well the data from numerical experiments.

In the limiting case of slow celerities $c \ll c_0$, as far as the mass and pseudomomentum are concerned, the dynamics of monotone shapes of 6GBE appears to be Newtonian in this limit since

$$M \approx M_0 \, , \quad P \approx Mc$$

The Kawahara solitons

The shapes of the sub-luminous solitary waves of 6GBE can transform to damped oscillatory ones by changing the coefficient β of the fourth-order dispersion. Increasing β one reaches a threshold above which the localized waves acquire oscillatory tails (called Kawahara solitons). The said threshold is usually a negative value, so that if one takes $\beta > 0$ the shapes will be Kawahara solitons for the whole range of admissible sub-luminous celerities. So here we report the case $\beta = 1$. There is a principal difference in this case from the previous one. Now the existence of the quasi-particles is not limited by the characteristic speed of the equation, but rather it is $c_0^2 = 0.75$, $c_0 \approx 0.866$ (see [78]).

Once again we found a best-fit approximation guided by the above described considerations. The result reads

$$M = M_0\gamma^{\frac{7}{4}}, \quad M_0 = 2.986, \qquad P = Mc\gamma^{\frac{5}{4}} = M_0 c\gamma^{\frac{12}{4}} \equiv M_0 c\gamma^3. \quad (4.6.13)$$

Here also the selected type of approximation secures a quantitatively very good result for the best fit.

There are some differences in the powers of γ between the two cases considered here. Yet, the general behavior is similar. One is to expect different behaviors from a complex system when one changes the sign of one of the dispersion coefficients. In Kawahara's case the two dispersions act against each other and this can explain the different shapes (damped and oscillatory) for the solitary waves and hence the different powers of γ in the expressions for the *mass* and *pseudomomentum*. The Newtonian limit obviously is $M \approx M_0, \; P \approx M_0 c$.

Example 11: Two-degrees of freedom generalized Boussinesq system

When the small coupling between the v degree of freedom of the previous example and the longitudinal displacement u and elongation strain $e = u_X$ is kept [79] — but remember that the ferroelastic phase-transition is driven through s, e being only a secondary subsystem —, then we have the following more complex system:

$$\begin{aligned} s_{tt} - c_T^2 s_{XX} + (s^3 - s^5 + 2\gamma s e + \alpha s_{XX})_{XX} &= 0 \, , \\ e_{tt} - c_L^2 e_{XX} + \gamma (s^2)_{XX} &= 0 \, , \\ s = v_X \, , \quad e = u_X \, , \quad (v, u) &\in \mathsf{R}^2 \, , \end{aligned} \qquad (4.6.14)$$

where γ is a coupling coefficient and c_L is a second characteristic speed. This system looks formidable. Still it admits exact analytical solitary-wave solutions which do represent the various transitions between austenite and two martensite variants. The sixth-order space derivatives have been discarded. With the quadruplet (s,q,e,r) we can rewrite system (4.6.14) as the *Hamiltonian system*

$$\begin{aligned} s &= q_{XX}\,, \quad e_t = c_L r_X\,, \\ q &= c_T^2 s - s^3 + s^5 - 2\gamma s e - \alpha s_{XX}\,, \\ r_t &= c_L e_X - \frac{\gamma}{c_L}(s^2)_X\,. \end{aligned} \tag{4.6.15}$$

And the associated total "mass", "momentum" and "energy" are given by

$$M = \int_{\mathsf{R}} s\mathrm{d}X = [v]_{-\infty}^{+\infty}, \tag{4.6.16}$$

$$P = \int_{\mathsf{R}} (sq_X + c_L er)\mathrm{d}X\,, \tag{4.6.17}$$

$$\begin{aligned} E = \int_{\mathsf{R}} \Big\{ &\frac{1}{2}(q_X^2 + c_L^2 r^2) + \frac{1}{2}(c_T^2 s^2 + c_L^2 e^2) - \gamma e s^2 \\ &-\frac{1}{4}S^4 + \frac{1}{6}S^6 + \frac{\alpha}{2}(s_X)^2 \Big\}\,\mathrm{d}X. \end{aligned} \tag{4.6.18}$$

With the definition (4.6.17), which we let the reader check that it is nothing but the canonical one with *ad hoc* conditions at infinity, we obtain a global balance of wave momentum in the inhomogeneous form

$$\frac{dP}{dt} = \mathcal{F} \equiv -\frac{1}{2}\left[\alpha s_X^2 + c_L^2 e^2\right]_{-\infty}^{+\infty}\,, \tag{4.6.19}$$

and this shows what conditions should apply to have conservation of wave momentum strictly enforced if we numerically work on a finite interval. This will be exploited in examining numerically the *solitonic* behavior of solutions of system (4.6.14) – see the works of Christov and Maugin [81] with numerous numerical results for the sole Boussinesq equation, and [41] where the interactions of *inelastic sech solutions* is exhibited for an "improved" version of the equation (see Example 12 for the notion of "improved" BO equation).

All above given examples are directly derived from a long-wave limit of the lattice dynamics of various elastic crystals. The next example is derived from *fluid mechanics* but it will be re-interpreted as an *elastic system* in our general field-theoretic context.

Example 12: Regularized long-wave Boussinesq (RLWBO) equation

It was remarked by several authors that in order to obtain good *stability properties*, it might be advisable to replace the fourth-order space derivative in the BO equation by a mixed *fourth-order space-time* derivative [55].

This is an alternate way to that proposed by the authors (Example 10) who rather introduce higher-order space derivatives. The result is the so-called *regularized long-wave Boussinesq* (RLWBO) *equation* that we may propose here in the form

$$u_{tt} - [u + F(u) + \beta u_{tt}]_{XX} = 0 \,, \tag{4.6.20}$$

where $F(u)$ is a polynomial in u starting with u^2. Introducing q, we can replace (4.6.20) by the *Hamiltonian system* [82]

$$\begin{aligned} u_t &= q_{XX} \,, \\ q_t - bq_{tXX} &= u + F(u) \,. \end{aligned} \tag{4.6.21}$$

This admits solitary-wave solutions. But the mixed space-time derivative in (4.6.20) creates some difficulty in interpretation for the following quantities. Mass, momentum and energy of soliton-like solutions are given by

$$\begin{aligned} M &= \int_{\mathsf{R}} u \mathrm{d}X \,, \quad P = -\int_{\mathsf{R}} u(q_X - \beta q_{XXX}) \mathrm{d}X \,, \\ E &= \int_{\mathsf{R}} \frac{1}{2}\left(u^2 + q_X^2 - 2\mathcal{U}(u) + \beta u_t^2\right) \mathrm{d}X \,. \end{aligned} \tag{4.6.22}$$

It is difficult to recognize in P and E the canonical definitions of such quantities. However, if we set $u = \bar{u}_X$ and note that $q_X = \bar{u}_t$, then the generalized kinetic energy is

$$\kappa = \frac{1}{2}(q_X^2 + \beta u_t^2) = \frac{1}{2}(\bar{u}_t^2 + \beta \bar{u}_{Xt}^2) \,, \tag{4.6.23}$$

and thus P is none other than

$$P = -\int_{\mathsf{R}} \bar{u}_X \frac{\delta \mathcal{L}}{\delta \bar{u}_t} \mathrm{d}X \,, \tag{4.6.24}$$

with a Lagrangian density

$$\mathcal{L} = \kappa - \left(\frac{1}{2}\bar{u}_X^2 + \mathcal{U}(\bar{u}_X)\right) \,, \tag{4.6.25}$$

where the functional Euler–Lagrange derivative is given by [39]

$$\frac{\delta \mathcal{L}}{\delta \bar{u}_t} = \frac{\partial \mathcal{L}}{\partial \bar{u}_t} - \frac{\partial}{\partial X}\left(\frac{\partial \mathcal{L}}{\partial \bar{u}_{Xt}}\right) = \bar{u}_t - \beta \bar{u}_{XXt} = q_X - \beta q_{XXX} \,. \tag{4.6.26}$$

Equation (4.6.24) is the canonical (field-theoretic) definition of the wave momentum when the inertial terms contain the strange field $\bar{u}_{Xt}$. The above scheme can be given a mechanical interpretation if we remember that $\bar{u}$ may be thought of as an elastic displacement for a one-dimensional rod model. Then β represents the so-called *lateral inertia* [10, 39] (remember that the Love–Rayleigh equation for rods accounting for lateral inertia reads: $u_{tt} - u_{XX} - bu_{ttXX} = 0$).

Note that while mass and energy are automatically conserved, momentum P in general satisfies a Newtonian equation of motion with pseudo force $\mathcal{F}$ on a finite spatial interval $[a, b]$ [82]:

$$\frac{dP}{dt} = \mathcal{F} \equiv -\frac{1}{2}\left[q_X^2 - \beta q_{XX}^2\right]_a^b . \tag{4.6.27}$$

Example 13: Nonlinear Maxwell–Rayleigh (NLMW) equation

This is a one-dimensional, obviously nonexactly integrable, equation obtained while studying the anomalous dispersion in a composite model that considers linear elastic resonators embedded in a nonlinear elastic matrix [39]. It reads

$$u_{tt} - [u + F(u) + \beta u_{tt}]_{XX} + \beta u_{tttt} = 0, \quad \beta > 0 . \tag{4.6.28}$$

Based on the experience of case (4.6.20), this is equivalent to the *Hamiltonian system*

$$\begin{aligned} u_t &= q_{XX} , \\ q_t - b(q_{tXX} - q_{XXX}) &= u + F(u) . \end{aligned} \tag{4.6.29}$$

The mass, momentum and energy are then defined by

$$\begin{aligned} M &= \int_{\mathrm{R}} u \mathrm{d}X , \\ P &= \int_{\mathrm{R}} u \left[q - \beta(q_{Xtt} - q_{XXX})\right] \mathrm{d}X , \\ E &= \int_{\mathrm{R}} \frac{1}{2}\left(u^2 + q_X^2 - \beta q_{Xt}^2 - 2\mathcal{U}(u) + \beta u_t^2\right) \mathrm{d}X . \end{aligned} \tag{4.6.30}$$

With $u = \bar{u}_X$ it can be checked that E and P have a canonical form with the general definition (4.6.24) for P for the potential $\bar{u}$ and local "kinetic energy" and Lagrangian given by

$$\begin{aligned} \kappa &= \frac{1}{2}\left(\bar{u}_t^2 + \beta(\bar{u}_{Xt}^2 - \bar{u}_{tt}^2)\right) , \\ \mathcal{L} &= \kappa - \left(\frac{1}{2}\bar{u}_X^2 + \mathcal{U}(\bar{u}_X)\right) . \end{aligned} \tag{4.6.31}$$

Again, the conservation of P over a finite interval would raise a problem. Furthermore, here the interpretation of the kinetic energy $(4.6.31)_1$ in terms of a simple mechanical system is not obvious. This is left for further work.

The last two examples have shown the crucial role played by the *kinetic energy* of the quasi-particle motion in the definition of the wave momentum. The next example is still more curious in that respect.

Example 14: Shallow-layer Boussinesq problem [78]

This is a two-degrees-of-freedom, one-dimensional problem. Let ψ and η represent the two relevant functions issued from fluid mechanics. We have the following consistent (from the point of view of conservation properties) model system:

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial X}\left(\eta \frac{\partial \psi}{\partial X}\right) &= -\frac{\partial^2 \psi}{\partial X^2} - \frac{\beta}{3}\frac{\partial^4 \psi}{\partial X^4} - \frac{2\beta^2}{15}\frac{\partial^6 \psi}{\partial X^6}, \\ \frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial X}\right)^2 &= -\eta. \end{aligned} \tag{4.6.32}$$

Introducing auxiliary variables q and u by $q_X = -h$ and $u = \psi_X$, we can rewrite the system (4.6.32) as

$$\begin{aligned} q_t + u q_X &= u + \frac{\beta}{3} u_{XX} + \frac{2\beta^2}{15} u_{XXXX}, \\ u_t + u u_X &= q_{XX}, \end{aligned} \tag{4.6.33}$$

where a double space integration has been performed and the boundary conditions $q_X = \eta = 0$ have been enforced at the lower end of the integral of integration $X = -L_1$. System (4.6.32) has an interesting but intriguing look. On an infinite interval of integration, i.e., R, for solitary-wave solutions we would impose the *asymptotic conditions*

$$\eta = -q_X \to \eta_\pm, \quad u = \psi_X \to 0, \quad \text{as} \quad X \to \pm\infty. \tag{4.6.34}$$

As a consequence, one also has the decay of all derivatives

$$\eta_X, \eta_{XX}, \cdots \to 0 \quad \text{and} \quad \psi_{XX}, \psi_{XXX}, \cdots \to 0 \quad \text{for} \quad X \to \pm\infty. \tag{4.6.35}$$

The situation is different for a *finite* interval of integration $[-L_1, +L_2]$ because depending on the set of boundary conditions considered we may or may not have conservation of "mass" and "energy" on a finite interval. The *conserving* boundary conditions that are compatible with the *physical* boundary conditions are

$$\psi_X = \psi_{XXX} = \psi_{XXXXX} = 0 \quad \text{at} \quad X = -L_1, +L_2. \tag{4.6.36}$$

The energy functional of the system (4.6.32) reads

$$E = \frac{1}{2}\int_{-L_1}^{L_2} \left(\eta^2 + \eta\psi_X^2 + \psi_X^2 - \frac{\beta}{3}\psi_{XX}^2 + \frac{2\beta^2}{15}\psi_{XXX}^2\right) \mathrm{d}X. \tag{4.6.37}$$

A possibility of *nonlinear blow-up* of the *solution* exists due to the presence of the term $\eta\psi_X^2$ which may happen to be negative for certain transients [83, 84].

What is the *wave momentum* of system (4.6.32)? Let us assume [78] that it is given by the expression

$$P = \int\limits_{-L_1}^{L_2} \eta\psi_X \mathrm{d}X\,. \tag{4.6.38}$$

The global balance law satisfied by this scalar quantity is obtained by multiplying eqn.(4.6.32)$_1$ by ψ_X, adding to this the X-derivative of (4.6.32)$_2$ multiplied by ψ_X, and integrating the whole over the finite interval, finally resulting in the *inhomogeneous* equation of "motion"

$$\frac{dP}{dt} = \mathcal{F} \equiv \frac{1}{2}\left[\frac{\beta}{3}\psi_{XX}^2 - \eta^2\right]_{-L_1}^{L_2}\,. \tag{4.6.39}$$

In terms of u and q, we can rewrite eqns.(4.6.38), (4.6.39) and (4.6.37) as

$$P = -\int_{-L_1}^{L_2} uq_X \mathrm{d}X\,, \quad \frac{dP}{dt} = \mathcal{F} = -\left[\frac{\beta}{3}u_x^2 - \frac{1}{2}q_X^2\right]_{-L_1}^{L_2}, \tag{4.6.40}$$

$$E = \int_{-L_1}^{L_2} \frac{1}{2}\left\{q_X^2 - q_X u^2 + u^2 - \frac{\beta}{3}u_X^2 + \frac{2\beta^2}{15}u_{XX}^2\right\}\mathrm{d}X\,. \tag{4.6.41}$$

Due to the boundary conditions (4.6.36) most of the terms will vanish and the only source of pseudoforce $\mathcal{F}$ could be the difference $\eta_+^2 - \eta_-^2$ of the fluid levels (in the original physical problem) which drives the then *unsteady wave*. Only when this difference vanishes can the *stationary* waves propagate at all. But *quid* of a purely *elastic re-interpretation*? If we note that the energy (4.6.37) or (4.6.41) is associated with the *Lagrangian density*

$$\mathcal{L} = \kappa(\psi_t;\psi_X) - W(\psi_X, \psi_{XX}, \psi_{XXX})\,, \tag{4.6.42}$$

where

$$\kappa = \frac{1}{2}(\psi_t^2 + \psi_X^2\psi_t)\,, \quad W = \frac{1}{2}\left\{\psi_X^2 - \frac{1}{4}\psi_X^2 - \frac{\beta}{3}\psi_{XX}^2 + \frac{2\beta^2}{15}\psi_{XXX}^3\right\},$$

then the general definition (4.6.24) and appropriate boundary conditions yield the canonical wave momentum

$$P = -\int\limits_{L_1}^{L_2} \psi_X \frac{\delta\mathcal{L}}{\delta\psi_t}\mathrm{d}X = -\int\limits_{L_1}^{L_2} \psi_X(\psi_t + \frac{1}{2}\psi_X^2)\mathrm{d}X\,, \tag{4.6.43}$$

which in effect is (4.6.38) on account of $(4.6.32)_2$, while the Euler–Lagrange equation derived from (4.6.42) by straightforward variation just yields the variant of $(4.6.32)_1$ by taking $(4.6.32)_2$ into account. Furthermore, the field-theoretical system obtained is of the *gyroscopic type* in the sense that the Lagrangian density (4.6.42) contains a term linear in ψ_t, so that this does not contribute to the *energy*, while it *does* contribute to the definition of canonical momentum (remember the above remark concerning the case of elastic ferromagnets). The numerical simulations of the above system are to be found in Christov,Maugin and Velarde [78], when the η variable has been eliminated in Boussinesq's manner, and the problem then reduces essentially to Example 10.

As the last detailed example we consider the G-Z system.

Example 15: Generalized Zakharov (GZ) system

This system which generalizes both the NLS equation (4.4.32) and the Zakharov system (4.4.36) of plasma physics was obtained while studying the possible propagation of *bright-envelope solitons* in shear-horizontal waves coupled to the *Rayleigh surface mode* on top of a mechanical structure made of a thin linear elastic film perfectly glued on a nonlinear elastic half-space (a certain type of *nonlinear waveguide* [85]). In the appropriate scaling it reads ($a \in \mathsf{C}, n \in \mathsf{R}$):

$$\begin{aligned} ia_t + a_{XX} + 2\lambda|a|^2 a + 2an &= 0\,, \\ n_{tt} - c_0^2 n_{XX} + \mu(|a|^2)_{XX} &= 0\,, \end{aligned} \tag{4.6.44}$$

where we recall that $n = u_X$, where u is an elastic displacement. The system (4.6.44) admits exact analytical solitary-wave solutions of the *bright-soliton* type for a accompanied by a solitary wave in n, but it obviously is *not* exactly integrable as both the nonlinearity parameter λ and/or the coupling between a and the *nondispersive subsystem* n destroy exact integrability. The method of *global conservation laws* to study further the properties of such a system is thus essential. Following along the Examples 7 and 8 above and noting the additional character of some canonical definitions, the "mass", momentum and energy of soliton-like solutions will be of the form (compare to eqns.(4.4.33)–(4.4.35) and (4.4.38))

$$M = \int_{\mathsf{R}} |a|^2 \mathrm{d}X\,, \quad P = \int_{\mathsf{R}} \{i(aa_t^* - a^* a_t) - u_X u_t\}\,\mathrm{d}X. \tag{4.6.45}$$

and

$$E = \int_{\mathsf{R}} \frac{1}{2}\left\{|a|^2 - \lambda|a|^4 - 2u_X|a|^2 + \mu^{-1}(u_t^2 + c_T^2 u_X^2)\right\}\mathrm{d}X \tag{4.6.46}$$

Note that M, here the *number of surface phonons* in the specific physical problem at hand, or the *total wave action*, is the same as for the NLS equation and for the Z system, while P is obviously the same as for the Z system.

The system (4.6.44) was studied in detail in Maugin et al. [84] and Hadouaj et al. [87]. In particular, for the exact solitary-wave solution given by these authors, the integration of (4.6.45) provides the looked for relationship between momentum P, the "mass" M, and the speed c of the quasi-particles associated with such localized-wave solutions, in the form

$$P(M,c) = Mc + \frac{2}{3}\mu M^3 c \left\{ \frac{\lambda + \mu(c_0^2 - c^2)^{-1}}{(c_0^2 - c^2)^2} \right\} \tag{4.6.47}$$

in which we identify a typical "Newtonian" contribution Mc, which is the one obtained for a pure NLS equation, and a non-Newtonian, non-Lorentzian contribution due to the μ coupling. For small cs this quasi-particle almost behaves like a Newtonian particle, but, assuming $\mu > 0, \lambda > 0$, there exists a *window* in speeds between c_0 and $c^+ \equiv \{c_0^2 + (\mu/\lambda)\}^{1/2}$ for which no propagation is possible, the "Newtonian" behaviour being recovered for high-speed "solitons". The P vs. c curve for this case is schematically represented in Figure 4.1. Note that for large amplitudes of the soliton, with increasing P, there are only one, three, four, three, and two possible wave speeds. This is a very strange *point-mechanics* indeed. The perturbation equation of the type $(4.5.6)_2$ can be exploited when there is further perturbation in the system (4.6.44). Dissipation in the first component is not a drastic event. Perturbation due to *viscosity* in the sub-system governing n, and resulting in a term γn_{XXt} in the right-hand side of $(4.6.44)_2$ is much more interesting. Equation $(4.5.6)_2$ then yields

$$\frac{dP}{dt} = \mathcal{F}_v \equiv \frac{4\gamma}{\mu} \int_{\mathsf{R}} n_X n_t \mathrm{d}X \,. \tag{4.6.48}$$

Using the known unperturbed one-soliton solution to evaluate the right-hand side for small viscosity γ, this leads to the differential equation

$$\frac{dP}{dt} = \mathcal{F}_v(M,c) = -\frac{3\gamma\mu M^5 c}{20} \left\{ \frac{\lambda + \mu(c_0^2 - c^2)^{-1}}{(c_0^2 - c^2)^2} \right\} . \tag{4.6.49}$$

For a certain range of initial speeds and "masses" (amplitudes), the qualitative discussion of (4.6.49) on account of the curve in Figure 4.1, shows that a new scenario of solitonic behavior is exhibited, a so-called *perestroïka* of the solution, which was corroborated by numerical simulations [88, 89], and shows that in such *nonlinear systems* a *weak dissipation* may cause a *violent rearrangement* of the dynamical solutions. This is enough to emphasize the interest for the global canonical conservation laws of solitonic systems. Let us note further some other systems of interest.

Example 16: Coupled GZ-BO systems

As a natural extension of the models of Examples 3 and 15 we note the following system which may also appear in the nonlinear elastic problem of surface waves on a crystal ($a \in \mathsf{C}, n \in \mathsf{R}$):

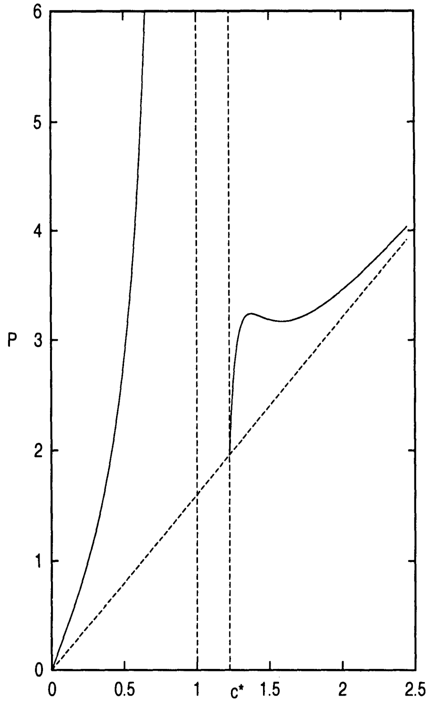

Figure 4.1: GZ system ($\lambda = 1$, $\mu = 0.5$, $c^* = 1.225$ and sufficiently large mass $M = 1.6$.

$$\begin{aligned} 2ia_t + 3a_{XX} + 2\lambda|a|^2 a - na &= 0\,, \\ n_{tt} - c_0^2 n_{XX} + (|a|^2 + n^2 + n_{XX})_{XX} &= 0\,, \end{aligned} \tag{4.6.50}$$

for which, to our knowledge, exact analytic one-soliton solutions are known only in the $\lambda = 0$ case. Thus the above technique can be used to study the influence of the *self-nonlinearity* with coefficient λ.

Examples 17: Nonlinear waves in rods

Depending on the complexity of the description of the lateral and cross-sectional deformation of the rod, a multitude of nonlinear dispersive models can be obtained [10, 90]–[93]. Typically these one-dimensional one-degree-of-freedom models are of the following types:

$$u_{tt} - u_{XX} - [F(u) + N(u_{XX}) + Q(u_{tt})]_{XX} = 0, \tag{4.6.51}$$

and

$$\Box_0 u - [F(u) + N(\Box_\alpha u)]_{XX} = 0\,, \tag{4.6.52}$$

where F is a polynomial in u starting with second degree, N and Q are linear combinations of successive even-order space and time derivatives starting with second-order ones, and $\Box_0$ and $\Box_\alpha, \alpha = 1, 2, \ldots$ are d'Alembertian linear wave operators. Examples 3, 5, 12 and 13 belong in this class. But more interesting for further developments of the presented method are true *two-dimensional* localized wave problems.

Example 18: Two-dimensional localized nonlinear waves

As an example we quote the following type of PDE which may arise in studying the two-dimensional (but one-degree of freedom) out-of-plane shear motion of an elastic surface [10]

$$\begin{aligned} u_{tt} &- (u_{XX} + u_{YY}) - \alpha\left[(\Phi u_X)_X + (\Phi u_Y)_Y\right] \\ &- \beta_1(u_{XXXX} + u_{YYYY}) - \beta_2 u_{XXYY} = 0, \end{aligned} \tag{4.6.53}$$

where u is an elastic displacement, $\Phi = u_X^2 + u_Y^2$ is a scalar invariant of isotropic elasticity, and α, β_1 and β_2 are three coefficients. Equations of the type (4.6.53) have also recently appeared in Kovalev and Syrkin [94]. While "energy" is still a scalar, momentum P now is a two-dimensional vector of components $-u_X u_t$ and $-u_Y u_t$ integrated over R^2, and its two-dimensional law of conservation can be exploited in the analytical and numerical study of the interactions of noncollinearly travelling localized wave solutions in R^2 (imagine localized humps circulating on the plane as on a deformable rubber sheet). In the same vein we mention the two-dimensional NLS obtained in certain systems of lattice models of elastic crystals in the long-wave approximation [95]

$$ia_t + p_1 a_{XX} + p_2 a_{YY} + q|a|^2 a = 0\,. \tag{4.6.54}$$

4.7 Conclusions

If we remember that many, if not all, of the systems that accompany the above developments find their origin in a discrete, *lattice dynamics* description of deformable solids, then we clearly witness a dialectical movement between the *discrete* and the *continuum* which can be illustrated in the flow chart in Figure 4.2.

Through the Ariane thread of Lagrangian–Hamiltonian formalism of the equations of the long-wave limit of discrete equations, and the canonical formulation of *conservation laws*, it is possible to establish "pseudo-Newtonian" equations of motion of certain solutions of complicated systems of PDEs as if these solutions behaved as *quasi-particles.* In these equations of motion the relationship between "momentum", "mass" and speed define a *point-mechanics* of a type that depends on the starting PDE or system of PDEs. Figure 4.3 illustrates this richness and originality: Figure 4.3(a) corresponds to a "Newtonian" relationship (starting from the NLS equation),

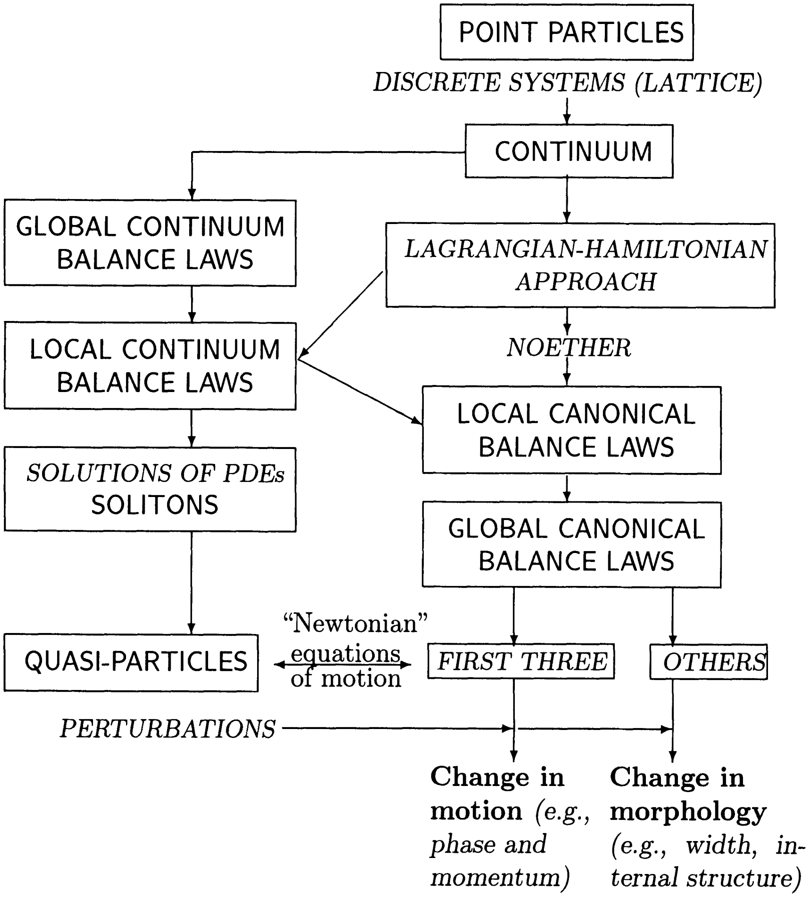

Figure 4.2: From particles in physical space to quasi-particles on the material manifold [13].

Figure 4.3(b) to a "Lorentzian–Einsteinian" relationship (starting from the SG equation), Figure 4.3(c) to the new mechanics obtained starting from the GZ system, and Figure 4.3(d) to the point mechanics associated with the 6GBO equation (see formulas (4.6.12), (4.6.13). Because of the limited number of attributes classically granted to point-particles, the *global* canonical conservation laws can only describe the crudest *changes* in motion of such "*wavicles*". From soliton theory we also know that additional conservation laws accompany exactly integrable systems, but we can only surmise that these will possibly describe the *internal changes* of such "*wavicles*", perhaps their morphology, in the same way as, in continuum mechanics we have come to endow a *material point* with an internal structure which is rigid in the simplest case (e.g., in *Cosserat* or *micropolar* continua [36, 39])

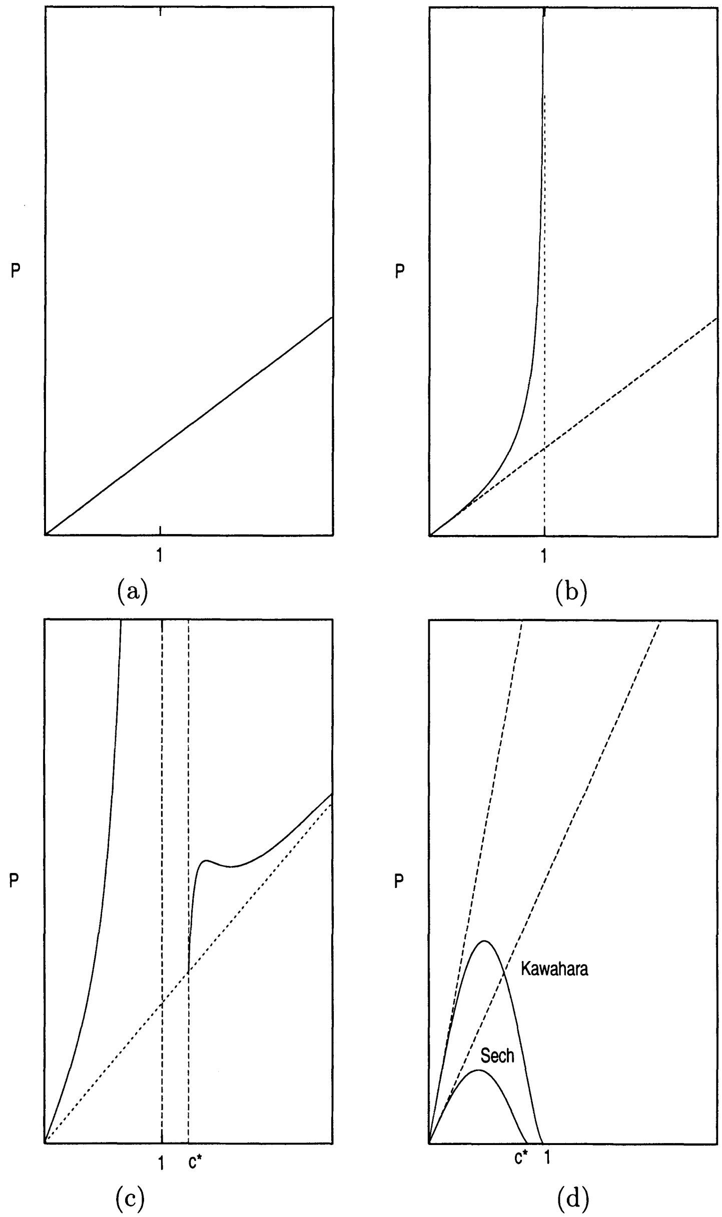

Figure 4.3: Schematic of (P, c) relationship for remarkable systems: (a) Nonlinear Schrödinger equation (Newtonian); (b) sine-Gordon equation (Lorentzian); (c) Generalized Zakharov system (compare Fig. 4.1); 6GBO equation (pseudo-Lorentzian dynamics).

or itself deformable in the more complicated case of so-called *micromorphic continua* [1]. We thus contemplate a description of complex nonlinear wave phenomena as *Matryoshka* (Russian-puppet box) nested constructs in which one recovers at one stage an internal structure proper to the previous stage. The difference perhaps resides in the fact that these different-level interpretations do not take place on the same *manifold* as the quasi-particles exhibited in this chapter do, just like the discrete lattices we can start with, in fact evolving on the *material manifold* $\mathcal{M}^3$ (according to the general description of Maugin [8]) while the local continuum balance laws have components on the physical manifold, the good old Euclidean space E^3 of physics or, if necessary, the Einsteinian arena of relativistic space-time physics V^4. Furthermore, just as the energy equation is the basic ingredient in measuring the performance of a numerical scheme (e.g., in *finite-difference* [97] and *spectral* [98, 99] methods), the additional conservation law of *pseudo-* or *canonical*, or else *wave*, momentum is another asset to assess the quality of such schemes and/or the nearly solitonic behavior of wave systems. The effort is to be placed on two- or three-dimensional systems which become more accessible and, in fact only accessible, with more powerful computing devices.

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