An Operator Splitting Scheme for Biharmonic Equation with Accelerated Convergence

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Abstract. We consider the acceleration of operator splitting schemes for Dirichlet problem for biharmonic equation. The two fractional steps are organized in a single iteration unit where the explicit operators are arranged differently for the second step. Using an *a-priori* estimate for the spectral radius of the operator, we show that there exists an optimal value for the acceleration parameter which speeds up the convergence from two to three times. An algorithm is devised implementing the scheme and the optimal range is verified through numerical experiments.

1 Introduction

Biharmonic boundary value problems arise in many different areas of mechanics of continua such as the stream function formulation for stationary Navier-Stokes equations and the equations for deformation of elastic plates. Constructing efficient numerical algorithms is of prime importance in these cases. A well established approach to the problem is to introduce an artificial time in the elliptic equation and to use operator splitting technique for the resulting parabolic equation. The technique is summarily known as Alternative Directions Implicit (ADI) method. To the authors' knowledge, the first work in which an ADI scheme was applied to biharmonic equation is [5]. The CD scheme exhibits the best of the world of ADI schemes, as being absolutely stable and low cost per iterations, but its rate of convergence has been shown to be rather slow in some cases (see, [7, 6]). In order to accelerate the convergence we consider an iteration unit consisting of a CD scheme and a modified scheme, the latter dependent on a parameter. The two schemes damp different part of the spectrum differently and we show in the present paper that the combination of them yields a faster convergence than the original ingredients.

2 Differential Equations and Conte-Dames ADI Scheme

Consider the following two-dimensional higher-order parabolic equation with Dirichlet boundary value problem

$$u_t = -\Delta^2 u(x,y) + F(x,y), \qquad (x,y) \in D; \tag{1}$$

$$u(0,y) = f_1(y), u(1,y) = f_2(y), u(x,0) = f_3(x), u(x,1) = f_4(x), \tag{2}$$

$$u_x(0,y) = g_1(y), u_x(1,y) = g_2(y), u_y(x,0) = g_3(x), u_y(x,1) = g_4(x), \tag{3}$$

I. Lirkov, S. Margenov, and J. Waśniewski (Eds.): LSSC 2005, LNCS 3743, pp. 387–394, 2006.

where D is a square region $\{(x,y)|0 < x < 1, 0 < y < 1\}$ and $\bar{D} = D \cup \partial D$ is its closure. Δ^2 is the biharmonic operator. Equations (1)-(3) are also called "clamped plate problem" in elasticity.

In order to obtain second order of approximation of the difference scheme we assume that function u possesses derivatives up to sixth order.

We employ an uniformly spaced mesh in $\bar{D}$ with spacings $h_x = h_y = h = 1/N$, where N is the grid size and replace D and ∂D by sets of grid points $D_h = \{(ih, jh)|i = 1, 2, \cdots, N-1; j = 1, 2, \cdots, N-1\}$ and ∂D_h. Respectively, $u^n_{i,j}$ is the difference approximation to u at the grid point $x = ih$, $y = jh$ and time stage $t = n\tau$, where τ is the time increment. The differential operators are approximated by the usual central difference operators, denoted by δ^4_x, δ^4_y and $\delta^2_x\delta^2_y$. The scheme of Conte and Dames [5] (called in what follows "CD") consists of following two sweeps in x and y directions

$$\tilde{u}_{i,j} = u^n_{i,j} - \tau(\delta^4_x\tilde{u} + 2\delta^2_x\delta^2_y u^n_{i,j} + \delta^4_y u^n_{i,j} - F_{i,j}), \tag{4}$$

$$u^{n+1}_{i,j} = \tilde{u}_{i,j} - \tau(\delta^4_y u^{n+1}_{i,j} - \delta^4_y u^n_{i,j}), \tag{5}$$

where $u^0_{i,j} = 0$ is an (arbitrary) initial condition and the time increment τ plays the role of an iteration parameter and can be chosen to accelerate convergence. Second-order approximations for the boundary conditions are obtained by means of central differences on the grid that overflows the actual region, e.g.

$$\tilde{u}_{i,j} = u^{n+1}_{i,j} = f_{i,j}, \qquad \text{for } (ih, jh) \in \partial D_h, \tag{6}$$

$$\phi_{-1,j} = \phi_{1,j} - 2hg_1(jh), \qquad \phi_{N+1,j} = \phi_{N-1,j} + 2hg_2(jh), \tag{7}$$

$$\phi_{i,-1} = \phi_{i,1} - 2hg_3(ih), \qquad \phi_{i,N+1} = \phi_{i,N-1} + 2hg_4(ih), \tag{8}$$

for $i = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, N-1$. Here ϕ stands for the different time stages $\tilde{u}$, and u^{n+1}.

Eliminating the intermediate variable $\tilde{u}_{i,j}$ between Eqs. (4) and (5) one gets

$$(E + \tau\delta^4_x + \tau\delta^4_y + \tau^2\delta^4_x\delta^4_y)u^{n+1} = (E - 2\tau\delta^2_x\delta^2_y + \tau^2\delta^4_x\delta^4_y)u^n + \tau F, \tag{9}$$

where F stands for the grid function of $F(x,y)$ on D_h, E denotes the identity matrix and the difference operators are considered as the corresponding matrices for grid function u^n. The subscripts i,j are omitted for brevity of notation. The transition matrix T from one time step to another is

$$T = (E + \tau\delta^4_x + \tau\delta^4_y + \tau^2\delta^4_x\delta^4_y)^{-1}(E - 2\tau\delta^2_x\delta^2_y + \tau^2\delta^4_x\delta^4_y). \tag{10}$$

The convergence of CD scheme for arbitrary positive iteration parameter τ is shown in [5] by demonstrating that $\|T\| < 1$ for any $\tau > 0$ for the case when the second-order derivatives are specified at the boundary. However,the dependence of spectral radius of T on the iteration parameter τ has not been investigated so far, because of the difficulty in obtaining appropriate eigenvectors of the difference operator for biharmonic problem with Dirichlet boundary conditions. If an arbitrary τ is chosen, the convergence rate of CD scheme can be quite slow (see, discussions in[7] and [6]).

Therefore we have three essential objectives to achieve in this work: (1) to reformulate the CD scheme in a manner that allows one to accelerate its convergence rate depending on an iteration parameter; (2) to prove that acceleration is possible and to find an estimate for the iteration parameter introduced; (3) to find optimal choice of the iteration parameter through numerical experiment.

Suppose that w is the grid function that is the solution of the stationary difference problem with the same Dirichlet boundary conditions. Define the error vector for the n-th iteration as $\xi^n = u^n - w$. By (9) and b.c. (6)-(8), we obtain the system of equations with homogeneous boundary conditions for ξ^n

$$\xi^{n+1} = T\xi^n \quad \text{on} \quad D_h, \tag{11}$$

$$\xi^{n+1} = 0 \quad \text{on} \quad \partial D_h, \tag{12}$$

$$\xi^{n+1}_{-1,j} = \xi^{n+1}_{1,j} \qquad \xi^{n+1}_{N+1,j} = \xi^{n+1}_{N-1,j}; \tag{13}$$

$$\xi^{n+1}_{i,-1} = \xi^{n+1}_{i,1} \qquad \xi^{n+1}_{i,N+1} = \xi^{n+1}_{i,N-1}. \tag{14}$$

Using Courant's theorem [2], Conte and Dames [5] showed that for the boundary conditions (12)-(14) the transition matrix T has a complete set of eigenvectors v_k for the vector space Φ defined on D_h and the corresponding eigenvalues are $0 < \alpha(T) = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{m-1} \leq \lambda_m = \beta(T) < 1$. This is the only a-priori information about transition matrix T needed for us to improve the original CD method. For convenience, we denote the matrices involved in the scheme as

$$B_x = E + \tau\delta_x^4, \qquad B_y = E + \tau\delta_y^4. \qquad R = E - 2\tau\delta_x^2\delta_y^2 + \tau^2\delta_x^4\delta_y^4. \tag{15}$$

Then T can be written as $T = B_y^{-1}B_x^{-1}R$.

3 The Modified Splitting Scheme

In order to accelerate the convergence of the iterations for a given time increment τ we try to find a modification of the CD scheme in which the transition operator has smaller norm than the original scheme.

The gist of present paper is that we introduce an iteration unit consisting of two CD iterations with different arrangements of the explicit terms by means of an auxiliary parameter θ as follows

$$\text{(n+1)-th unit} = \begin{cases} B_xB_yu^{n+1} = Ru^n + \tau F & (a) \\ B_xB_yu^{n+2} = R[(\theta+1)u^{n+1} - \theta u^n] + \tau F & (b) \end{cases}, \tag{16}$$

where the intermediate variable $\tilde{u}$ is eliminated and $(\theta+1)u^{n+1} - \theta u^n$ is considered as the input vector to compute u^{n+2} using CD scheme in (b). The boundary conditions are the same for the two steps and are omitted for the sake of brevity. Clearly, there is no extra cost for implementing of the new scheme.

To study the convergence of the iteration units we observe that one can recast the equations of the full iteration unit (16) as equations for the error ξ^n, namely

$$\xi^{n+1} = T\xi^n, \quad \xi^{n+2} = [(\theta+1)T^2 - \theta T]\xi^n. \tag{17}$$

Begin with a trivial initial condition $u^0 = 0$, which means that the initial condition for the error is $\xi^0 = -w$. The Fourier expansion of ξ with respect to the complete set of eigenvectors $\{v_k\}$ of T reads

$$\xi^0 = -w = \sum_{k=1}^{m} c_k v_k, \qquad \xi^n = \begin{cases} \sum_{k=1}^{m} \lambda_k \rho_\theta^l(\lambda_k) c_k v_k & \text{if } n = 2l+1 \\ \sum_{k=1}^{m} \rho_\theta^l(\lambda_k) c_k v_k & \text{if } n = 2l \end{cases} \tag{18}$$

where c_k is the corresponding Fourier coefficient for $l = 0, 1, 2, \cdots$; where we define quadratic function $\rho_\theta(\lambda_k) = (1+\theta)\lambda_k^2 - \theta\lambda_k$ and λ_k is the corresponding eigenvalue for T. In order to show the convergence of the iteration units, it is sufficient for us to show that for each λ_k, $|\rho_\theta(\lambda_k)| < 1$, since $0 < \lambda_k < 1$. Indeed, we have

$$\max_{\lambda_k} |\rho_\theta(\lambda_k)| \leq \max_{0<x<1} |\rho_\theta(x)| \leq \max\{|\rho_\theta(0)|, |\rho_\theta(1)|, |\rho_\theta\left(\frac{\theta}{2(\theta+1)}\right)|\}, \tag{19}$$

where we consider only $\theta > 0$ because only positive choice of θ can accelerate the iterations. Since $\rho_\theta(0) = 0$ and $\rho_\theta(1) = 1$, Eq.(19) will hold when

$$|\rho_\theta\left(\frac{\theta}{2(\theta+1)}\right)| = \frac{\theta^2}{4(\theta+1)} < 1, \tag{20}$$

whence it follows that $0 < \theta < 2 + 2\sqrt{2}$ must hold in order to ensure that we have $\max_{\lambda_k} |\rho_\theta(\lambda_k)| < 1$ needed for the convergence of the iteration unit (16).

Now we analyze the acceleration of the convergence for the iteration units. Consider the error ξ_{cd}^n for the CD scheme after n iterations. Using (11) and (18) we can represent ξ_{cd}^n in terms of Fourier expansion with respect to the eigenvectors v_k of the transition matrix T, namely

$$\xi_{cd}^n = \sum_{k=1}^{m} \lambda_k^n (c_k v_k). \tag{21}$$

The coefficients of v_k in the error vector ξ_{cd}^n decrease in absolute value by the multiplicative factor of λ_k. The least affected is the coefficient of v_m, which corresponds to the largest eigenvalue λ_m. Repeating CD iterations leads us to the asymptotic case, because for $n \gg 1$ all other coefficients become negligibly small compared to the coefficient of v_m, and hence we have the error for CD scheme given by

$$\xi_{cd}^n = \lambda_m \xi_{cd}^{n-1} = \beta(T)\xi^{n-1}. \tag{22}$$

From (22) follows for the standard norm $\|\xi_{cd}^n\|^2 = (\xi_{cd}^n, \xi_{cd}^n)$ that

$$\|\xi_{cd}^n\| = \beta(T)\|\xi_{cd}^{n-1}\|. \tag{23}$$

Although for small n the amplifier of the norm depends on the iteration, the performance of the iterative process is usually judged by the asymptotic rate of convergence s ($n \gg 1$) which is defined as

$$s = -\ln|\beta(T)| = -\ln\frac{\|\xi^n\|}{\|\xi^{n-1}\|}. \tag{24}$$

By the recursive relation (23) we obtain the asymptotic equation

$$\|\xi^n\| = e^{n \ln \beta(T)} \|\xi^0\| = e^{-ns} \|\xi^0\|. \tag{25}$$

In this way, the asymptotic rate of convergence s characterizes the rate of the exponential error decrease. The cause of slow convergence of CD scheme is that $\beta(T)$ is close to unity. In such a case we use the notation $\beta = 1 - p$ where $0 < p \ll 1$. Using the Taylor expansion, the asymptotic rate of convergence s of CD scheme is given by $s = -\ln(1-p) \approx p \ll 1$.

In the same manner, we investigate the asymptotic rate of convergence s of our iteration units. In order not to obscure the main idea we limit the discussion here to some typical values of θ, as $\theta = 3$, which is in the range $0 < \theta < 2 + 2\sqrt{2}$. The coefficient of eigenvector v_k in (18) decreases in absolute value by the multiplicative factor of $|\rho_3(\lambda_k)|$. Since $\rho_3(x) = 4x^2 - 3x$ is a quadratic function, it is easy to show that its minimum is at $x = \frac{3}{8}$ and has the magnitude of $\frac{9}{16}$. Then

$$\max_{0<\lambda_k\le 1-p} |\rho_3(\lambda_k)| \le \max\{\tfrac{9}{16}, |\rho_3(1-p)|\}, \tag{26}$$

which mean that if $9/16 > |\rho_3(1-p)|$, we have already obtained a very fast convergence of the iteration units which will reduce the error to 10^{-5} within 20 iteration units.

On the other hand, for $p \ll 1$ we have $|\rho_3(1-p)| = |(1-p)(1-4p)| \approx (1-5p)$ and using (17) asymptotically for $n \gg 1$, we can write $\|\xi^{n+2}\| \approx (1-5p)\|\xi^n\|$. Therefore, the corresponding asymptotic rate of convergence of our iteration units is

$$s = -\tfrac{1}{2} \ln(1-5p) \approx 2.5p, \tag{27}$$

where we compare one iteration unit consisted of two iterations with two original CD iterations for which the reduction factor would be $(1-p)^2 \approx 1-2p$. All this means that the introduction of the iteration unit can speed up the convergence rate at least 2.5 times. Note that the actual factor of acceleration depends on the value of time increment τ and is discussed in the next Section. Similarly, we can verify that for $0 < \theta < 2 + \sqrt{2}$, the asymptotic rate of convergence is

$$s = -\tfrac{1}{2} \ln \rho_\theta(1-p) = -\tfrac{1}{2} \ln\{(1-p)[(\theta+1)(1-p) - \theta]\} \approx \tfrac{1}{2}(\theta+2)p. \tag{28}$$

But this conclusion depends on the assumption that $1 - \beta(T) = p \ll 1$ and

$$\left| \rho_\theta \left(\frac{\theta}{2(1+\theta)} \right) \right| = \frac{\theta^2}{4(1+\theta)} \le \rho_\theta(1-p). \tag{29}$$

Actually, the maximum eigenvalue $\beta(T)$ of the transition matrix can be estimated in the numerical experiment, based on which we can choose a proper value of the auxiliary parameter θ to maximize the acceleration of the devised iteration units. By (9) and (10), we have the recursive relation

$$(u^{n+1} - u^n) = T(u^n - u^{n-1}). \tag{30}$$

Since $\|T\| = \beta(T) < 1$, after a few CD iterations the largest eigenvalue $\beta(T)$ becomes the dominant multiplicative factor in the iterations while other smaller eigenvalues become negligible. Therefore we can compute the numerical quantity

$$q = \|u^{n+1} - u^n\| / \|u^n - u^{n-1}\| \tag{31}$$

in each iteration and when q varies little between two consecutive iterations, an estimate of $\beta(T)$ is obtained. Based on the a-posteriori numerical estimate $q \approx \beta(T)$, we can determine the optimal choice of θ to maximize the acceleration. At first, we notice that $\rho_\theta(q)$ is a linear function of θ when q is fixed

$$h_1(\theta) = \rho_\theta(q) = (q^2 - q)\theta + q^2. \tag{32}$$

since $q^2 - q < 0$, $h_1(\theta)$ is a monotone decreasing function. Next, we consider another function h_2 of θ

$$h_2(\theta) = \left|\rho_\theta\left(\frac{\theta}{2(1+\theta)}\right)\right| = \frac{\theta^2}{4(1+\theta)}, \tag{33}$$

which is a monotone increasing for $\theta \in (0, 2\sqrt{2})$ because

$$h_2'(\theta) = \frac{\theta^2 + 2\theta}{4(\theta+1)} > 0. \tag{34}$$

To maximize the acceleration, by (19) we need to find θ from

$$\min_{0<\theta<2+\sqrt{2}} \max\{|h_1(\theta)|, h_2(\theta)\}. \tag{35}$$

By the monotonicity of $h_1(\theta)$ and $h_2(\theta)$, it is easy to verify that the optimal θ has to be the positive solution of the following equation and automatically in the range $(0, 2+2\sqrt{2})$,

$$(q^2 - q)\theta + q^2 = \tfrac{1}{4}\theta^2(1+\theta)^{-1}. \tag{36}$$

Therefore, we obtain the optimal θ by solving (36)

$$\theta_{opt}(q) = \frac{2q^2 + (\sqrt{2}-1)q}{-2q^2 + 2q + \frac{1}{2}}. \tag{37}$$

Since $h_1(0) = q^2$ and $h_1(\theta)$ is a decreasing function, then we have

$$0 < h_1(\theta_{opt}) = \frac{\theta_{opt}^2}{4(1+\theta_{opt})} < q^2 \tag{38}$$

where q^2 stands for the convergent effect of two CD iterations. Hence, we have shown that for different choices of the iteration parameter τ in the CD method, which give us different transition matrices T (i.e. different q), we are always able to select a θ_{opt} to accelerate the original CD iterations.

By (28) and (37), we can see that when $\beta(T)$ is closer to unity, that is to say, the original CD method has a slower convergence, larger auxiliary parameter θ can be chosen in the iteration units which leads us to a more significant acceleration over the CD method. Since the value of θ is bounded by $2 + 2\sqrt{2} \approx 4.8$, by (28) the best acceleration that our method can reach is 3.4 times faster than CD method.

4 Results and Discussion

We begin with numerical verification of the performance of a single iteration-unit as introduced above. In implementation of the algorithm we follow [4] where a splitting scheme of type of CD was applied to lid-driven cavity flow of viscous liquid. Later on a similar algorithm was used in [3] for another kind of higher-order diffusion equation.

We select two different problems for which analytical solutions are available

$$\hat{u}(x,y) = \sin^2(\pi x)\sinh^2 y, \ (a) \quad \text{and} \quad \hat{u}(x,y) = 2350x^4(x-1)^2y^4(y-1)^2 \ (b). \tag{39}$$

The first of these was created by us, and the secon is from [1].

The operator to be inverted is the same in both cases and does not depend on the actual solution, and hence on the right hand side. Yet, the convergence rate depends on the specific solution because of the fact that different eigenfunctions decay differently with the iterations. Since different analytical solutions have different content of eigenfunctions, the rate could be rather different which warrants checking the performance for two radically different analytical solutions.

It is important to demonstrate that an iteration unit does give the same truncation error as a single CD iteration. For this particular test we fix $\theta = 1, 2$ and $\tau = \frac{1}{4}h$ and calculate the solution with three different grids. Here, we fix the value of θ and vary the number of grid points both along x and y directions simultaneously. We define the computed order of approximation as $R = \log_2 [\|u_N - \hat{u}\|/\|u_{2N} - \hat{u}\|]$. For second order schemes $O(h^2)$, the value of $R = 2$. Table 1 shows that in both cases, the computed convergence rate is very close to two. Having confirmed the second-order accuracy of the scheme we can address the issue of computational efficiency. The pertinent parameter here is the number of iterations, say N_{iter}, needed to reduce the norm of the difference between two iterations to 10^{-6}. Clearly, the rate of convergence is a function of the time increment, τ, and the optimization parameter θ. For different θ we performed calculations with several different τ. In Fig. 1 we present N_{iter} for which the norm between two iterations for Eq. (39)(b) go down to 10^{-6}. For $\theta \leq 2$ there is a conspicuous minimum of N_{iter} for the original CD scheme, as well as for the scheme with one iteration unit. In this range of θ, our N_{iter} is at least twice smaller than CD scheme. In the range of non-optimal τ, the acceleration is even larger. The result in Fig. 1 is in very good agreement with the theoretical estimate, Eq. (37), which gives us $\theta_{optimal} \approx 1.8$ when $q \approx 0.77$. This is the fastest result one can get for Eq. (39)(b) with a splitting scheme of the type of CD.

Table 1. Order of approximation R for different grids

case Eq. (39)(a)

N	$\theta = 1$	R	$\theta = 2$	R
256	1.578E-5	-	1.582E-5	-
512	3.797E-6	2.05	3.810e-6	2.05
1024	9.687E-7	1.97	9.747E-7	1.967

case Eq. (39)(b)

N	$\theta = 1)$	R	$\theta = 2$	R
256	3.093E-4	-	3.080E-4	-
512	7.981E-5	1.95	8.063e-5	1.93
1024	1.823E-5	2.13	1.884E-5	2.09

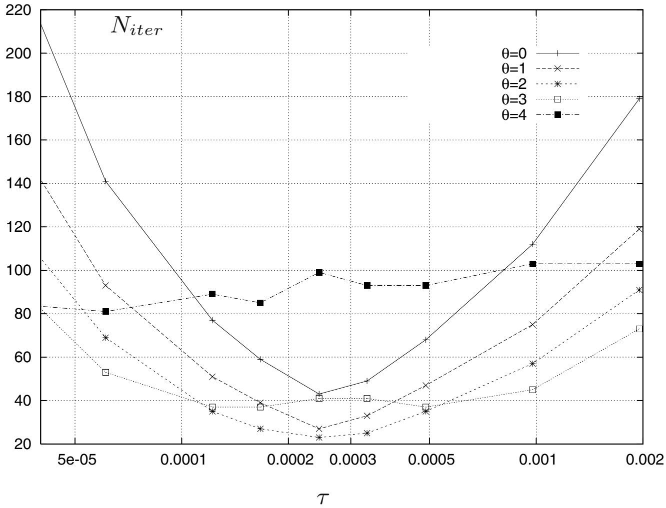

Fig. 1. Number of iterations, as function of time increment for different values of θ and grid size 1024×1024

It is interesting to mention here the nonmonotone behavior of N_{iter} with the increase of θ, which means that there is an optimum for θ, but only in the vicinity of optimal τ. If one cannot chose *a-priori* an optimal τ, then the scheme proposed here will overperform the CD scheme even for a wider range of θ. We obtained a similar result for grid size 512×512, for which the values of N_{iter} consistently lower by 10% from the presented case. This is natural for iterative algorithms since the eigenvalues of difference operators depend on spacing h.

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