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POISSON-WIENER EXPANSION IN STATISTICAL THEORY OF DILUTE SUSPENSIONS

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In recent years, the development of the adequate statistical description of stochastic systems has constantly been in the focus of scientific attention, because of its outstanding importance for a number of physical applications such as turbulence, heterogeneous flows and materials, random noises, etc. It was Wiener [1] who adapted to the case of random functions the idea of Volterra concerning the representing a functional in series with respect to a given function. Since Wiener employed the Gaussian random white noise as a basis function, the Hermite polynomials emerged in the series and the method was named "Wiener-Hermite expansion" (see for details [2]). The Wienet-Hermite method has been applied to a variete of stochastic problems and has proved to be a powerfull tool for attacking them.

However, the very essence of this method - the employment of Gaussian white noise as a basis function - makes the Wiener-Hermite expansion not quite fitted to the stochastic systems with dilute random nonhomogeties because the Gaussianity implies a very dense population of very small nonhomogenities. Naturally, this is not the case in most of stochastic physical systems since their nonhomogenity approximates ususally the Poisson random function. Poisson function has been introduced into Wiener series in [3], where the name "Poisson-Wiener expansion" has been coined. The outstanding role played by the latter in modelling of physical systems, however, has been only recently reveiled in [4,5] where the required technique for attacking the nonlinear i.e. the ensemble average of density is just the so-called suspension mean density. Turning to the velocity ensemble average one obtains

$$(5)\qquad \langle\vec{v}\rangle = \vec{v}_0 + \gamma \iiint_{-\infty}^{\infty} \vec{K}(\vec{x}-\vec{\xi})\,d^3\vec{\xi} = \vec{v}_0 + c(v_1 - v_0) + (1-c)\vec{v}^{*},$$

where $\vec{v}_1$ denotes the average velocity in the particle and $\vec{v}^{*}$ - the average of the diturbancy field outside the particle. The latter can be specified only after solving the problem for kernel $\vec{K}$. It is to be mentioned that ensemble average $\langle\vec{v}\rangle$ of velocity coincides with the so-called mean-volume velocity of suspension.

The following average properties of Poisson functions are needed for further manipulations

$$\langle f(\xi)\rangle = \gamma$$

$$(6)\qquad \langle f(\xi_1) f(\xi_2)\rangle = \gamma\,\delta(\xi_1-\xi_2) + \gamma^2$$

$$\langle f(\xi_1) f(\xi_2) f(\xi_3)\rangle = \gamma\,\delta(\xi_1-\xi_2)\,\delta(\xi_1-\xi_3) + \gamma^2\left[\delta(\xi_1-\xi_2) + \delta(\xi_2-\xi_3) + \delta(\xi_1-\xi_3)\right] + \gamma^3$$

(for details see [3,4,5]). Making use of the above properties one obtains the following equation of continuity

$$(7)\qquad \frac{\partial\langle\rho\rangle}{\partial t} + \operatorname{div}\left[\langle\rho\rangle\langle\vec{v}\rangle\right] = -\operatorname{div}\left[(\rho_1-\rho_0)\,c\,(\vec{v}_1-\vec{v}_0)\right]$$

It is well seen that because of drop relative motion there arises a sourse term for the average continuum. Here one should be awere that $\langle\vec{v}\rangle$ is not a mean-mass velocity [6].

In the same manner an averaged system of equations is derived from the set of Navier-Stokes equations:

$$\frac{\partial\langle\rho\rangle\langle\vec{v}\rangle}{\partial t} + \frac{\partial}{\partial t}\left[c(\rho_1-\rho_0)(\vec{v}_1-\vec{v}_0)\right]$$

$$(8)\qquad + \nabla\cdot\left\{\langle\rho\rangle\langle\vec{v}\rangle\langle\vec{v}\rangle + 2(\rho_1-\rho_0)\,c\,\langle\vec{v}\rangle(\vec{v}_1-\vec{v}_0) + \langle\rho\rangle\gamma\iiint_{-\infty}^{\infty}\vec{K}^2(\vec{\xi})\,d^3\vec{\xi}\right\}$$

$$= \nabla\cdot\left\{\mu_0\nabla\vec{v}_0 + (\mu_1-\mu_0)\,c\,\nabla\vec{v}_0 + (\mu_1-\mu_0)\gamma\iiint_{-\infty}^{\infty} h(\vec{\xi})\,\vec{K}(\vec{\xi})\,d^3\vec{\xi}\right\}$$

It is easily seen that all the additional terms to the equations of pure liquid are of order of $O(c)$ or $O(\gamma)$. An amazing feature of (8) is the additional term with a time derivative occurring due to the particulate phase relative motion.

The set of equations (7) and (8) can be closed making use of the last of equalities (6). In order not to complicate much the calculations, only the slow relative motion problems has been devoloped.

The most important feature of Poisson-Wiener expansion is that each higher-order term contributes to the average characteristics a quantity which is of order of the respective power of the concentration of nonhomogenities. This means that a truncation of the series at a sertain degree forms an asymptotic expansion with respect to the concentration. Nothing like that is known for Wiener-Hermite expansion. In the present short note an application of Poisson-Wiener method to the case of dilute suspensions is outlined retaining all the terms of first-degree order with respect to concentration, i.e. the hydrodynamic fields are developed in truncated Poisson-Wiener series (see [5]) as follow:

$$(1) \qquad \Phi = \Phi_0 + \iiint_{-\infty}^{\infty} \Phi_1(\vec{x}-\vec{\xi}) f(\vec{\xi}) d^3\vec{\xi} + O(\gamma^2) \ ,$$

where $\Phi = \vec{U}, \rho, P, \mu$ and

$$(2) \qquad h(\vec{x}-\vec{\xi}) = \begin{cases} 1 & \text{inside the particle} \\ 0 & \text{outside the particle} \end{cases}$$

is the so-called characteristic function of particle (drop). In present note only a suspension of spherical particles of uniform radius is considered. Stochasticity comes through the random positions of these particles, which volume is

$$(3) \qquad V = \iiint_{-\infty}^{\infty} h(\vec{x}-\vec{\xi}) d^3\vec{\xi}$$

and the volume fraction (concentration) of the particulate phase is $C = \gamma V$. The other quantities can be specified as follows:

$\vec{U}_0, \rho_0, P_0, \mu_0$ - continuous-phase velocity, density, pressure and viscosity,

$\vec{U}_1, \rho_1, \mu_1$ - velocity of the center of particle, density and viscosity of fluid in drops.

Respectively, $\vec{K}(\vec{x}-\vec{\xi})$ and $P(\vec{x}-\vec{\xi})$ are the velocity end pressure fields created by the presence of a single particle in the flow of the continuous phase. Main object of the present note is to outline the way of specifying these functions.

It is obvious that

$$(4) \qquad \langle \rho \rangle = \rho_0 + (\rho_1 - \rho_0)\gamma \iiint_{-\infty}^{\infty} h(\vec{\xi}) d^3\vec{\xi} \equiv \rho_0 + (\rho_1 - \rho_0)\gamma V = \rho_0 + (\rho_1 - \rho_0) C \ ,$$

of particles will be considered, i.e. the nonlinear and non-steady terms in Navier-Stokes equations will be omitted.

What is obtained by multiplying equations of Navier-Stokes by the quantity $f(\vec{\delta})-\gamma$ and taking the ensemble averaging is :

$$
(9) \qquad \begin{aligned} O(\gamma^2) &= \gamma \nabla_\xi P + \gamma(\vec{f}_1-\vec{f}_0)h(\vec{\xi}) + \nabla_\xi \cdot \left\{ \left[\mu_0 + (\mu_1-\mu_0)h(\vec{\xi}) \right] \nabla_\xi \vec{K} \right\} \\ O(\gamma^2) &= div_\xi \vec{K} \, , \end{aligned}
$$

which is nothing else but the governing system of equations for a single-particle slow motion in an unbounded fluid, if only $\vec{K}$ is thought of as a velocity field created by the suspended particle. Solution of this problem is known [7] , i.e. the problem can be closed and the unknown quantities in (7) and (8) can be identified.

In conclusion it should be stressed that the equations obtained are valid only if $C^2 \ll 1$ and only if the lenght scale of the particle motion (namely, its radius a) is much smaller than the lenght scale defined by the velocity gradient of the averaged flow. Under these assumptions the Poisson-Wiener expansion forms a rigorous in asymptotic sence basis under the statistical theory of dilute suspensions.

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