LOCALIZED STRUCTURES AND SOLITARY WAVES EXCITED BY INTERFACIAL STRESSES

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ABSTRACT. Arguments and numerical evidence are provided for the appearance of solitary waves driven by surface tension tractions (Marangoni effect)

In Bénard convection [1,2] instability induced by the Marangoni effect [3] is expected to trigger and sustain not only steady convective patterns but also transverse and longitudinal waves[4-7]. In Refs. 6 & 7 a theory was provided for the appearance of solitary waves of Korteweg-de Vries form [8,9]. Here we provide albeit in a succint way further theoretical predictions and some numerical evidence that substantiates that theory (see also Refs. 10-12 for a more detailed account). For simplicity we consider an open shallow horizontal liquid layer, thickness d, subjected to a transverse vertical temperature gradient in a two-dimensional geometry. This is the simplest model for Bénard-Marangoni convection[1-3]. Let x and z denote the horizontal and vertical coordinates, respectively. With (u, w) the velocity components, ρ, p, T and σ the density, pressure, temperature and air-liquid interfacial tension, the evolution of the liquid layer is governed by the following set of equations [13-16]

$$u_x + w_z = 0 \tag{1}$$

$$\rho_0(u_t + uu_x + w\,u_z) = -p_x + \mu(u_{xx} + u_{zz}) \tag{2a}$$

$$\rho_0(w_t + uw_x + ww_z) = -p_z + \mu(w_{xx} + w_{zz}) - \rho g \tag{2b}$$

$$T_t + uT_x + wT_z = \kappa\,(T_{xx} + T_{zz}) \tag{3}$$

$$\rho = \rho_0[1 - \alpha\,(T - T_0)]\,,\ \alpha = \left(\frac{\partial\rho}{\partial T}\right)/\rho_0 \tag{4}$$

$$\sigma = \sigma_0 + \gamma(T - T_0)\,,\ \gamma = (\partial\sigma/\partial T) \tag{5}$$

where g denotes the gravitational acceleration, ρ_0 and σ_0 are some reference values, e.g. at the temperature T_0=T(z=d), and subscripts "t","x" and "z" refer to time and space derivatives. Note that generally α is positive (except water around 4°C) and γ though gener-

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ally negative may according to the liquid become positive[17,18]. μ is the dynamic viscosity and κ is the thermal diffusivity of the liquid.

The above given evolution equations must satisfy boundary conditions (b.c.):

- at $z = d+\eta$, with $\eta(x, t)$ the time and space dependent interfacial deformation

$$w = \eta_t + u\eta_x \tag{6}$$

$$p - p_{air} = \frac{2\mu}{N^2}[w_z + u_x\eta_x^2 - \eta_x(u_z+w_x)] - \frac{\eta_{xx}\sigma}{N^3} - \frac{\eta_{xx}}{N^5}(\bar{\kappa} + \bar{\varepsilon})(u_x + \eta_x w_x) \tag{7}$$

$$\mu(1-\eta_x^2)(u_z+w_x) + 2\mu\eta_x(w_z - u_x) = N\eta_x\sigma_z + N\sigma_x + \frac{(\bar{\kappa}+\bar{\varepsilon})}{N}(u_{xx} + \eta_x w_{xx}) + \frac{(\bar{\kappa}+\bar{\varepsilon})}{N^3}\eta_{xx}(w_x - \eta_x u_x) \tag{8}$$

$$k[T_z - \eta_x T_x] = -k\beta N - h N(T - T_0) \tag{9}$$

- at z=0

$$w=0 \tag{10}$$

$$u_z = 0 \quad \text{(stress-free)} \tag{11}$$

$$T=T_b \text{ ("ideal" conductor) or } T_z= \text{ constant ("insulator")} \tag{12}$$

The alternative to (11) is the more realistic case of a rigid boundary, u=0. However we take the stress-free case for simplificity. We have introduced the two shear and dilational surface viscosities, κ and ε, respectively. $N^2 = 1 + \eta_x^2$. The quantity h/k accounts for the air-liquid heat transfer characteristics, with $k=\kappa\rho c$ the thermal conductivity and c the specific heat . Later on we shall use the temperature gradient $\beta= (T_b - T_0)/d$, taken positive when the heating is from the liquid side as in the traditional Bénard set-up[1-3]. T_0 is a reference value taken at the undeformed liquid-air interface in the quiescent state and $T_b=T(z=0)$. Note that for simplicity we have considered the air above the liquid layer weightless and dynamically passive and at the pressure p_{air} coincident with the liquid´s value at z=d.

The motionless steady state denoted with subscript "s" is a solution of the above posed problem. We have

$$T_s(z) = T_b - \beta z \tag{13}$$

$$\rho_s(z) = \rho_0[1 + \alpha\beta(z-d)] \tag{14}$$

$$u_s=0 \tag{15}$$

$$w_s=0 \tag{16}$$

$$p_s(z) = p_{air} - \rho_0 g[(z-d) + \frac{\alpha\beta}{2}(z-d)^2] \tag{17}$$

As new *units* we choose "d" for length, βd for temperature, κ/d for velocity, d^2/κ for time and $\mu\kappa/d^2$ for pressure, respectively. For later convenience we also introduce the following dimensionless groups: $P=\mu/\kappa\rho_0$ (Prandtl number), $Bi=hd/\kappa$ (Biot number), $K=\mu\kappa/\sigma_0 d$ (Capillary number), $B=\rho_0 g d^2/\sigma_0$ (Bond/"Eötvös" number), $Vi=(\kappa+\varepsilon)/\mu d$ (Surface viscosity number), $R=\alpha g\beta d^4/\kappa\nu$ (Rayleigh number), $M=-\gamma\beta d^2/\kappa\nu$ (Marangoni number or surface velocity Reynolds number) with $\nu=\mu/\rho_0$, the kinematic viscosity.

For the the air-liquid interface a reasonable assumption is that the heat transfer is rather weak. Introducing ε, a smallness parameter, to a first approximation we can set $Bi = \varepsilon^2$. This estimate is consistent with a search of long wavelength excitations because these convective modes generally dissipate the least. For the remaining variables we shall use: $\xi=\varepsilon(x-Ct)$, $\tau=\varepsilon^3 t$, $C=C_0+\varepsilon^2 C_2+\ldots$, $u=\varepsilon^2(u_0+\varepsilon u_1+\ldots)$, $w=\varepsilon^3(w_0+\varepsilon w_1+\ldots)$, $p-p_s=\varepsilon^2(p_0+\varepsilon p_1+\ldots)$, $T-T_s=\varepsilon^3(\theta_0+\varepsilon\theta_1+\ldots)$, $\eta=\varepsilon^2(\eta_0+\varepsilon\eta_1+\ldots)$, $R=R_0+\varepsilon^2 R_2+\ldots$, $M=M_0+\varepsilon^2 M_2+\ldots$ and $Vi=\varepsilon^2(Vi_0+\varepsilon^2 Vi_2+\ldots)$.

Using these scalings, inserting the above given expansions in the evolution equations and b.c., and noticing that ε becomes an *ordering* parameter, from the original nonlinear problem we get a hierarchy of linear inhomogeneous equations that have to be sequentially solved subjected to a solvability condition (Fredholm alternative). Here in practice then latter reduces to integrating the horizontal Navier-Stokes equation over the liquid layer.

To the lowest order approximation, i.e. at the ε^0 order we get the already given basic steady state. Then at the following order of approximation in ε, the solvability condition provides $C_0=\{BP/K-M_0\}^{1/2}$ which shows the correction in the celerity (phase velocity) of the wave due to the Marangoni effect. In dimensional form its value is

$$C_0^2 = gd + \left[60\gamma\mu\kappa^2/(2\rho_0^2 g\kappa\alpha d^4+5\rho_0\kappa d^2\gamma)\right] \tag{18}$$

that for a *shallow* layer can be approximated by

$$(C_0^2)_{sw} \approx gd(1+12\mu\kappa/\rho_0 g d^3) \tag{19}$$

It shows a correction of order d^{-3} which differs from the *ideal, dissipation-freee* KdV where the correction scales with (1/d). For a $d=10^{-2}$ cm liquid layer of silicone oil our predicted correction due to the Marangoni effect is about 30%.

To the order ε^2, from the solvability condition we get the neutral stability locus

$$\frac{M_0}{-12} + \frac{R_0}{30} = 1 \tag{20}$$

that for vanishing buoyancy, $R_0=0$, reduces to $M_0= -12$. For a standard liquid with negative $\gamma=(\partial\sigma/\partial T)$, a negative Marangoni number demands that the *heating be done from the air side.* Qualitatively we recover the case for which earlier predictions were reported concerning on the one hand the onset of capillary-gravity and longitudinal waves in a deep, actually an infinitely deep liquid layer[4,5], and on the other hand KdV excitations in a shallow liquid layer subjected to Marangoni stresses[6,7].

The value $R_0 = 30$ at vanishing Marangoni effect has already been reported in the literature[19,20]. This critical Rayleigh number defines the onset of a KdV excitation in a buoyancy-driven Bénard-Rayleigh liquid layer heated from below and open to the ambient air, with no Marangoni stresses involved. Although for comparison with experiment the latter case seems quite an unrealistic it shows the *ubiquity* of the Korteweg-de Vries equation when the shallow water approximation is used.

The next term in the hierarchy of equations is at the ε^3 order. The solvability condition demands that the standard (ideal) Korteweg-de Vries equation[8,9] be satisfied. The modification due to the Marangoni effect and dissipation appears at the following order of approximation, i.e. at the ε^4 order. Then in the laboratory reference frame (x,t) we get

$$\eta_t + C_{exp}\eta_x + \alpha_1\eta\eta_x + \alpha_2\eta_{xxx} + \alpha_3\eta_{xx} + \\ + \alpha_4\,\eta_{xxxx} + \alpha_5\,(\eta\eta_x)_x = 0 \tag{21}$$

The coefficients C_{exp}, α_1, α_2 α_3, α_4 and α_5 are rather cumbersome expressions not needed here. Eq.(21) is our *dissipation-modified* Korteweg-deVries equation. It is a combination of the Kuramoto-Sivashinsky-Velarde equation[21,22] and the Korteweg-de Vries equation [8,9]. C_{exp} is the *experimentally measurable* velocity in the laboratory frame of reference. This velocity depends only on known or measurable quantities. As we do not expect confusion to arise in what follows we shall simplify notation by denoting C_{exp} with C alone.

We are interested in the localized solutions, e.g. solitary waves of the nonlinear equation (21). Going back to the moving coordinate frame η is function of the coordinate $\xi= x -Ct$ only, and then eq. (21) is reduced after integration to the following ODE

$$-C\eta+\alpha_1\eta^2+\alpha_2\eta_{\xi\xi}+\alpha_5\eta\,\eta_\xi+\alpha_3\eta_\xi+\alpha_4\eta_{\xi\xi\xi} = 0 \tag{22}$$

where the integration constant is set equal to zero because we consider only solutions for which $\eta(-\infty) = 0$. With apropriate rescaling of the solution [22] we get (22) even for $\eta(-\infty) \neq 0$.

At the other infinity $\eta \to +\infty$ the solution can approach zero (homoclinics) or the constant c/α_1 (heteroclinics/kinks). For sake of illustration and without aiming at completeness here we shall be concerned with the homoclinics of (22). Without loss of

generatity we can assume $\alpha_4 = 1$ since $\alpha_4 \neq 0$ and then the equation can be appropriately rescaled. It is convenient to rewrite (22) in *normal form* by setting

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -\alpha_3 y - \alpha_5 xy - \alpha_1 x^2 + cx - \alpha_2 z \end{aligned} \tag{23}$$

for which we have the inverse problem of finding solutions decaying on both infinities

$$x,y,z \to 0 \qquad \text{for} \quad \xi \to \pm\infty \tag{24}$$

This is typically an inverse problem where we have more boundary conditions than the order of the systems. In Ref. 23 was proposed the method of variational imbedding (VI) for solving this kind of problems and applied to the particular case of the Lorenz system. The method proved efficient and has permitted to obtain the solitary wave solutions of the Kuramoto-Sivashinsky [24] and Kuramoto-Sivashinsky-Velarde equations [11].

Following the procedure described in Refs. 11, 24 & 25 we consider the minimisation of the functional

$$\begin{aligned} I = \sum_{i=2}^{N-1} \Bigg\{ & \left(\frac{x_{i+1}^{n+1} - x_i^{n+1}}{h} - y_i^{n+1}\right)^2 + \left(\frac{y_{i+1}^{n+1} - y_i^{n+1}}{h} - z_i^{n+1}\right)^2 \\ & + \Bigg[\frac{z_{i+1}^{n+1} - z_i^n}{h} + \alpha_3 y_i^{n+1} + \alpha_5 y_i^{n+1} x_i^n + \alpha_2 z_i^{n+1} \\ & - c x_i^{n+1} + 2\alpha_1 x_i^n x_i^{n+1} - \alpha_1 (x_i^n)^2 \Bigg]^2 \Bigg\} \end{aligned} \tag{25}$$

where the nonlinear term is suitably quasi-linearized. h is the spacing, N the total number of grid points and L= (N-1)h is the "actual" infinity to which the infinite interval is reduced for the sake of difference approximation.

The minimization of (31) provides three tri-diagonal linear systems for the three unknown set (x_i, y_i, z_i), namely

$$\begin{aligned} & \frac{1}{h^2}\left(z_{i+1}^{n+1} - 2z_i^{n+1} + z_{i-1}^{n+1}\right) - \left(1+\alpha_2^2\right) z_i^{n+1} + \frac{1}{h} y_{i+1}^{n+1} - \frac{1}{h} y_i^n - \frac{\alpha_2}{h} z_{i+1}^{n+1} \\ & + \frac{\alpha_2}{h} z_i^{n+1} + \frac{1}{h}\left(\alpha_3 + \alpha_5 x_i^n\right) y_i^{n+1} - \frac{1}{h}\left(\alpha_3 + \alpha_3 x_i^n\right) y_{i-1}^{n+1} - \alpha_2\left(\alpha_3 - \alpha_3 x_i^n\right) y_i^{n+1} \\ & = \alpha_2\left[\alpha_1 (x_i^n)^2 - c x_i^n\right] - \frac{1}{h}\left[\alpha_3 (x_i^n)^2 - c x_i^n\right] + \frac{1}{h}\left(\alpha_i x_{i-1}^2 - c x_{i-1}^n\right) \end{aligned} \tag{26}$$

$$\frac{1}{h^2}(y_{i+1}^{n+1}-2y_i^{n+1}+y_{i-1}^{n+1}) - \left[1+(\alpha_3+\alpha_5 x_1^n)^2\right] y_i^{n+1} + \frac{1}{h}(x_{i+1}^{n+1}-x_i^{n+1})$$

$$\frac{1}{h}(z_i^{n+1}-z_{i-1}^{n+1}) - \frac{1}{h}(\alpha_3+\alpha_5 x_i^n)(z_{i+1}^{n+1}-z_i^{n+1}) - \alpha_2(\alpha_3+\alpha_5 x_i^n) z_i^{n+1} \tag{27}$$

$$= (\alpha_3+\alpha_5 x_i^n)\left[\alpha_1 (x_i^n)^2 - c x_i^n\right]$$

$$\frac{1}{h^2}(x_{i+1}^{n+1}-2x_i^{n+1}+x_{i-1}^{n+1}) - (\alpha_5 y_i^n - c + 2\alpha_1 x_i^n)^2 x_i^{n+1} \tag{28}$$

$$= \frac{1}{h}(y_i^{n+1}-y_{i-1}^{n+1}) + (\alpha_5 y_1^n - c + 2\alpha_i x_i^n)\left[\frac{1}{h}(z_{i+1}^{n+1}-z_i^{n+1}) + \alpha_2 z_i^{n+1} + \alpha_3 y_i^{n+1} - \alpha_1 (x_i^n)^2\right]$$

Discussing the technical details of application of the VI method is beyond the scope of this note. Suffices to mention that the difference scheme (32)-(34) is of first-order of approximation 0(h) but in the final results the accuracy is of order $0(h^2)$. This is obtained by means of Richardson extrapolation of the solution on different meshes (with diffrent number N of grid points, $N_1 = 3201$ and $N_2 = 6401$). It was already done in Refs. 24-26 where it was shown the usefulness and efficiency of the VI method to get higher order accuracy.

Rather than presenting here an exhaustive series of calculations for the complete set of governing parameters of (28) we shall illustrate some of the results found. For instance, the KSV case corresponds to $\alpha_2 = 0$ and α_5 variable. Figs. 1-4 depict some findings with α_5 called δ (a more complete account and further details about these findings can be found in Ref. 11).

To see the role of the term with coefficient α_2 we set $\alpha_5=0$ and $\alpha_1=3$ and $\alpha_3=1$ in order to comply with the earlier numerical investigation on KS [24] and KSV equations [11]. Thus we merely have a KS equation with a major component from the KdV equation. Indeed the value $\alpha_2=0$ corresponds to KS while $\alpha_2 >> 1$ should result in essentially a KdV behavior by appropriate rescaling and magnitude decrease of the coefficients of the KS part. In Fig. 5 we present the shape of the solitary wave obtained for different values of α_2 ranging from 0 to 10 and C=1216. It is clearly seen that the shape transforms from from a KS solitary wave with wavy forerunner to the standard KdV solitary wave shape. We have found that for the value $\alpha_2=10$ there is practically no difference with the KdV solution $-\mathrm{sech}^2(\xi)$. Even more we found that Bénard-Marangoni convection is dominated by the KdV dynamics already for $\alpha_2 > 3$ which is exactly the order of magnitude of α_1.

There is little experimental information on Marangoni-driven solitary waves. However, some twenty five year old work by Linde and collaborators on Marangoni-driven instabilities have recently been reviewed and evidence has been found of solitary waves. Ref. 26 provides an account of solitary wave signatures identified in Linde's work.

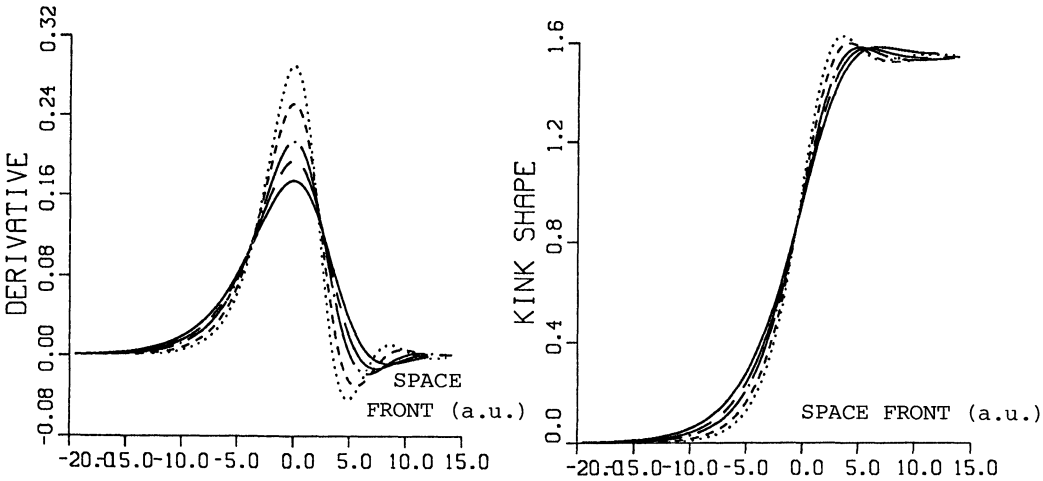

Fig.1 The localized solution of KSV equation for negative values of δ:
. . . δ=0: - - - δ=-0.1; — • — δ=-0.2; — - — δ=-0.3; ——— δ=-0.4.

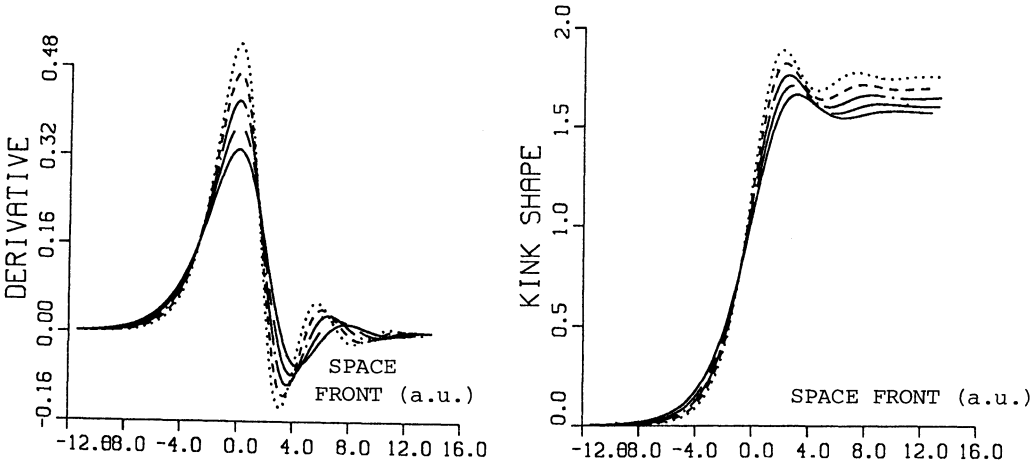

Fig.2 The localized solution of KSV equation for negative values of δ:
. . . δ=-0.5: - - - δ=-0.6; — • — δ=-0.7; — - — δ=-0.75; ——— δ=-0.8.

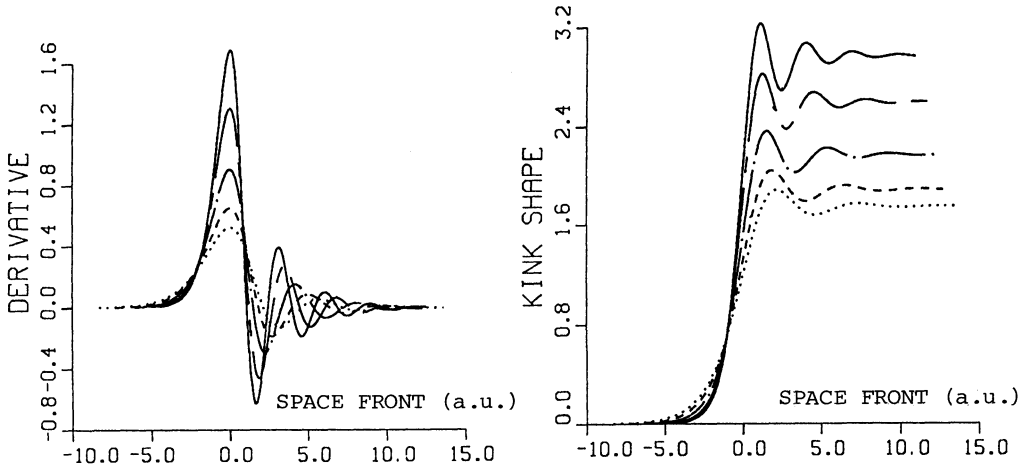

Fig.3 The localized solution of KSV equation for positive values of δ: . . . δ=0: - - - δ=0.2; — • — δ=0.5; — - — δ=0.8; ——— δ=1.0.

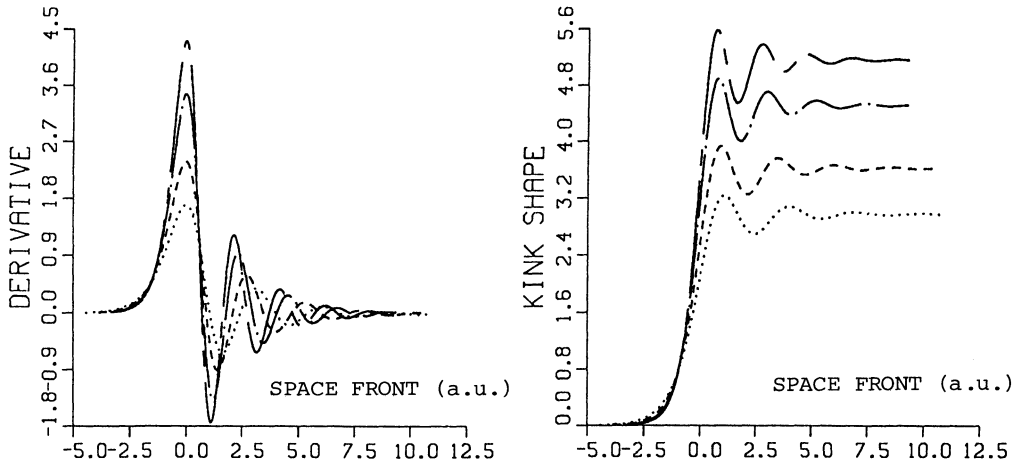

Fig.4 The localized solution of KSV equation for positive values of δ: . . . δ=1.0: - - - δ=1.2; — • — δ=1.4; — - — δ=1.5.

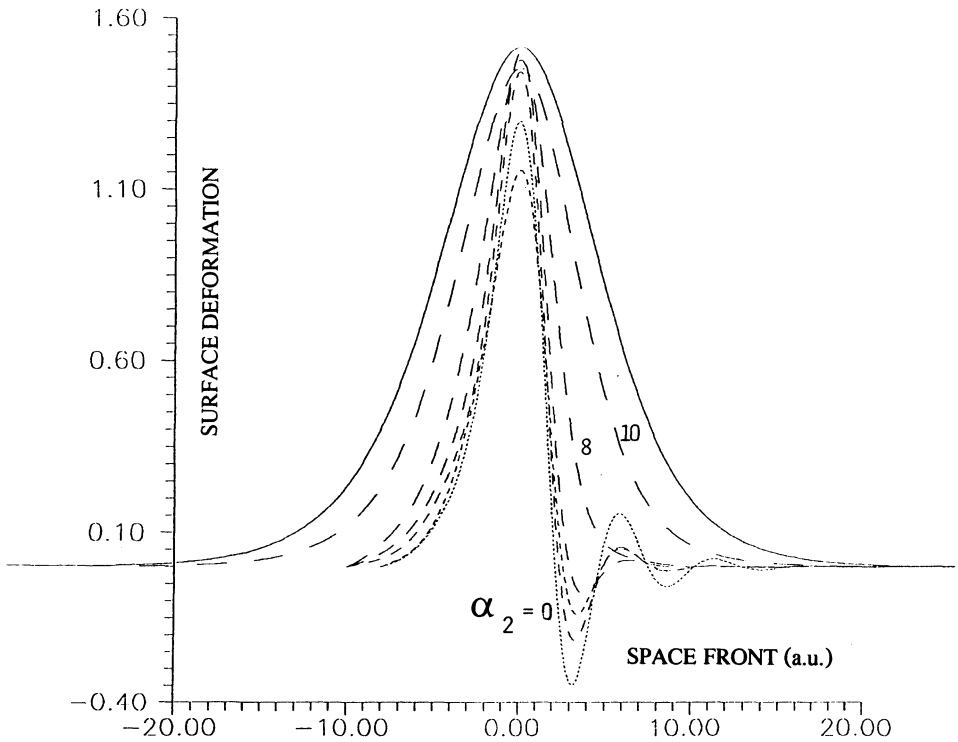

Fig. 5 The modification induced by the Marangoni effect upon the KdV solitary wave for different values of α_2 ranging from 0 to 10. The solid line corresponds to the standard KdV case.

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REFERENCES

1.E.L. Koschmieder, Adv. Chem. Phys. **26**, 177 (1971)
2.M.G. Velarde and C. Normand, Sci. Am. **243**, 92 (1980)
3.The role and relevance of the Marangoni effect in science and engineering is extensively described in several chapters of *Physicochemical Hydrodynamics: Interfacial Phenomena,* edited by M.G. Velarde (Plenum, New York, 1988)
4.M.G. Velarde and X.-L. Chu, Phys. Scripta **T 25**, 231 (1989)
5.X.-L. Chu and M.G. Velarde, Physicochem. Hydrodyn. **10**, 727 (1988)
6.X.-L. Chu and M.G. Velarde, Phys. Rev. **A 43**, 1094 (1991)
7.M.G. Velarde, X.-L. Chu and A.N. Garazo, Phys. Scripta **T 35**, 71 (1991)
8.G.B. Whitham, *Linear and Nonlinear Waves* (J. Wiley, N.Y., 1974)
9.A. Jeffrey and T. Kawahara, *Asymptotic Methods in Nonlinear Wave Theory* (Pitman, London, 1982)
10.A.N. Garazo and M.G. Velarde, Phys. Fluids **A** (submitted)
11.C.I. Christov and M.G. Velarde, Phys. Rev. **A** (submitted)
12.M.G. Velarde, A. N. Garazo and C.I. Christov, Phys. Rev. **A** (submitted)
13.R. Aris, *Vectors, Tensors and the Basic Equations of Fluid Mechanics* (Prentice Hall, Englewood Cliffs, N.J., 1962)
14.B. Levich, *Physicochemical Hydrodynamics* (Prentice Hall, Englewood Cliffs, 1962)
15.R. Pérez Cordón and M.G. Velarde, J. Phys. (Paris) **36**, 591 (1975)
16.M.G. Velarde and R. Pérez Cordón, J. Phys. (Paris) **37**, 177 (1976)
17.P.J. Desré and J.C. Joud, Acta Astronaut. **8**, 407 (1981)
18.J.C. Legros, M.C. Limbourg and G. Petré, Acta Astronaut. **11**, 143 (1984)
19.C.M. Alfaro and M.C. Depassier, Phys. Rev. Lett. **62**, 2597 (1990)
20.H. Aspe and M.C. Depassier, Phys. Rev. **A 41**, 3125 (1990)
21.J.L. Castillo, P.L. Garcia Ybarra and M.G. Velarde, in *Sinergetics and Dynamic Instabilities*, edited by G.Caglioti, H.Haken and L.Lugiato (North-Holland, Amsterdam, 1988)
22.J.M. Hyman and B. Nichols in *Partial Differential Equations,* edited by M.Crandall (Academic, New York, 1987)
23.C.I. Christov, in *Proceedings 14th Spring Conference, Union of Bulgarian Mathematicians* (Sunny Beach, Bulgaria, 1985)
24.C.I. Christov and V.P. Nartov, in *Proceedings Conference Anniversary Academician L. Tchakalov* (Samokov, Bulgaria, 1986) (in Russian)
25.C.I. Christov, J. Comp. Phys. (submitted)
26.P. Weidmann, H. Linde and M.G. Velarde, Phys. Fluids **A** (submitted)